

THE ALEXANDROV-TOPONOGOV COMPARISON THEOREM FOR RADIAL CURVATURE

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ABSTRACT. We discuss the Alexandrov-Toponogov comparison theorem under the conditions of radial curvature of a pointed manifold (M, o) with reference surface of revolution $(\widetilde{M}, \tilde{o})$. There are two obstructions to make the comparison theorem for a triangle one of whose vertices is a base point o . One is the cut points of another vertex $\tilde{p} \neq \tilde{o}$ of a comparison triangle in \widetilde{M} . The other is the cut points of the base point o in M . We find a condition under which the comparison theorem is valid for any geodesic triangle with a vertex at o in M .

1. INTRODUCTION

The Alexandrov-Toponogov comparison theorem (shortly ATCT) has been very useful in the study of geometry of geodesics including Riemannian geometry. In the present paper we discuss ATCT under the conditions of radial curvature of a pointed manifold (M, o) . We can see historical remarks and many references about ATCT in [2], [5].

Reference space is a surface of revolution $(\widetilde{M}, \tilde{o})$ with a geodesic polar coordinate system (r, θ) around \tilde{o} such that its metric is given by

$$ds^2 = dr^2 + f(r)^2 d\theta^2.$$

Let $p \in M$ be fixed and set $\tilde{p} = (d(o, p), 0) \in \widetilde{M}$. We may find a point $\tilde{q} \in \widetilde{M}$ satisfying

$$d(\tilde{o}, \tilde{q}) = d(o, q) \quad \text{and} \quad d(\tilde{p}, \tilde{q}) = d(p, q)$$

for any point $q \in M$. We call \tilde{q} the *reference point* of q and the map $\Phi : M \rightarrow \widetilde{M}$ given by $q \mapsto \tilde{q}$ the *reference map* of M into \widetilde{M} . It is

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not certain whether or not every point $q \in M$ has a reference point and every geodesic triangle $\triangle opq$, $p, q \in M$, admits the corresponding geodesic triangle $\triangle \tilde{o}\tilde{p}\tilde{q}$, $\tilde{p}, \tilde{q} \in \tilde{M}$.

There are two obstructions to ATCT. In the present paper we show that ATCT is established if the obstructions do not occur simultaneously at any point in M .

Let $T(p, q)$ denote a minimizing geodesic segment connecting p and q in M and $\tilde{T}(p, q) = \Phi(T(p, q))$. ATCT is valid if the reference curve $\tilde{T}(p, q)$ and the minimizing geodesic segment $T(\tilde{p}, \tilde{q})$ satisfy the good positional relation (see (2.2)). One obstruction to establish it is the appearance of cut points $Cut(\tilde{p})$ to \tilde{p} in the reference curve $\tilde{T}(p, q)$ of $\triangle opq$. In fact, we do not know what relation there is between the positions of the reference curve $\tilde{T}(p, q)$ and $T(\tilde{p}, \tilde{q})$ if the reference curve $\tilde{T}(p, q)$ intersects $Cut(\tilde{p})$. The cases treated in all papers for ATCT referred to a surface of revolution are free from the obstruction on the cut loci. Actually, it is proved in [3] that ATCT for any geodesic triangle with a vertex o holds if there are no cut points in a half part of \tilde{M} whose boundary consists of two meridians with angle π at the vertex \tilde{o} .

In general, the composition of the reference map Φ with r -coordinate function may have a local maximum. This may cause a bad positional relation between $\tilde{T}(p, q)$ and $T(\tilde{p}, \tilde{q})$. The other obstruction is the existence of the local maximum points of the distance function to o in M , which is restricted to an ellipsoid with foci at o and p . Let $E(p) \subset M$ be the set of all such local maximum points. Then, $E(p)$ is a subset of the cut points to the base point o (see Lemma 14). The reference points of $E(p)$ may be in the boundary of the image of the reference map of M locally, so there possibly exists a minimizing geodesic segment whose endpoints are reference points but containing a non-reference point in \tilde{M} . This situation shows that ATCT may not be true.

It is natural to ask what condition verifies the existence of a minimizing geodesic segment $T(\tilde{p}, \tilde{q})$ in \tilde{M} which has the good positional relation with $\tilde{T}(p, q)$. We assume that the reference points of all points in $E(p)$ are not cut points of \tilde{p} (see (2.1)). Under this assumption we prove that ATCT holds for any geodesic triangle $\triangle opq$.

We say that (M, o) is *referred* to (\tilde{M}, \tilde{o}) if every radial sectional curvature at $p \in M$ is bounded below by $K(d(o, p))$, where $K(t)$ is the Gauss curvature of \tilde{M} at the points in the parallel t -circle.

Theorem 1 (The Alexandrov convexity). *Assume that a complete pointed Riemannian manifold (M, o) is referred to a surface of revolution $(\widetilde{M}, \tilde{o})$. If a point $p \in M$ satisfies $\Phi(E(p)) \cap \text{Cut}(\tilde{p}) = \emptyset$, then for any geodesic triangle $\triangle_{opq} \subset M$ with $q \neq o, p$ there exists its comparison triangle $\triangle_{\tilde{o}\tilde{p}\tilde{q}} \subset \widetilde{M}$ such that $d(\tilde{o}, \tilde{x}) \leq d(o, x)$ for any point $\tilde{x} \in T(\tilde{p}, \tilde{q})$ and $x \in T(p, q)$ with $d(\tilde{p}, \tilde{x}) = d(p, x)$. Here $T(p, q)$ and $T(\tilde{p}, \tilde{q})$ are the bases of geodesic triangles \triangle_{opq} and $\triangle_{\tilde{o}\tilde{p}\tilde{q}}$, respectively.*

Theorem 2 (The Toponogov comparison theorem). *Under the same assumptions as in Theorem 1, every geodesic triangle $\triangle_{opq} \subset M$ admits its comparison triangle $\triangle_{\tilde{o}\tilde{p}\tilde{q}} \subset \widetilde{M}$ such that*

$$\angle_{opq} \geq \angle_{\tilde{o}\tilde{p}\tilde{q}}, \quad \angle_{oqp} \geq \angle_{\tilde{o}\tilde{q}\tilde{p}}, \quad \angle_{poq} \geq \angle_{\tilde{p}\tilde{o}\tilde{q}}.$$

As an application of these theorems we are allowed to define a pointed Alexandrov space (M, o) with radial curvature bounded below by a function K (see Remark 10 and sequent paragraphs).

Corollary 3. *Let (M, o) be a compact Riemannian manifold which is an Alexandrov space with a base point at o with radial curvature bounded below by the function K . Here $K : [0, \ell] \rightarrow \mathbb{R}$, $\ell < \infty$, is the radial curvature function of $(\widetilde{M}, \tilde{o})$. Then, the perimeters of all geodesic triangles \triangle_{opq} in M are less than or equal to 2ℓ and the diameter of M is less than or equal to ℓ . Moreover, if there exists a geodesic triangle \triangle_{opq} in M whose perimeter is 2ℓ , then M is isometric to the warped product manifold whose warping function is K . In particular, the same conclusion holds for M if the diameter of M is ℓ .*

The following corollary is proved in [4] when M is a noncompact pointed Riemannian manifold with radial curvature bounded below by the function K which is monotone non-increasing. We call such a surface of revolution with monotone non-increasing curvature function a *von Mangoldt surface*. There are no cut points in an open half part of a von Mangoldt surface \widetilde{M} whose boundary consists of two meridians with angle π at the vertex \tilde{o} (see [6]).

Corollary 4. *Let (M, o) be a noncompact Alexandrov space with a base point at o with radial curvature bounded below by the function K . Here $K : [0, \infty) \rightarrow \mathbb{R}$ is the radial curvature function of $(\widetilde{M}, \tilde{o})$. If the total curvature of \widetilde{M} is positive, then M has one end and has no straight line.*

These results will be stated more precisely in §2 after introducing some definitions and notations.

The idea of the proof of the theorems is this. The good positional relation (2.2) is equivalent to the Aleksandrov convexity and the Toponogov angle comparison (see Remark 6). Therefore, we study what positional relation holds between the reference curve $\widetilde{T}(p, q)$ of every minimizing geodesic segment $T(p, q)$ in M and a minimizing geodesic segment $T(\tilde{p}, \tilde{q})$ in \widetilde{M} . To do this we use the partial order \leq in the set of all curves which are parameterized by the angle coordinate θ in \widetilde{M} . Let $U(\tilde{p}, \tilde{q})$ and $L(\tilde{p}, \tilde{q})$ denote the minimizing geodesic segments connecting \tilde{p} and \tilde{q} in \widetilde{M} such that $L(\tilde{p}, \tilde{q}) \leq T(\tilde{p}, \tilde{q}) \leq U(\tilde{p}, \tilde{q})$ for any minimizing geodesic segment $T(\tilde{p}, \tilde{q})$, namely all minimizing geodesic segments $T(\tilde{p}, \tilde{q})$ lie in the biangle domain bounded by $L(\tilde{p}, \tilde{q}) \cup U(\tilde{p}, \tilde{q})$ in \widetilde{M} (see §6).

Let \widetilde{M}_p^+ denote the half part of \widetilde{M} bounded by the union of the meridians through \tilde{p} and opposite to \tilde{p} . For $r > d(o, p)$ we define an ellipsoid in M by $E(o, p; r) = \{x \in M \mid d(o, x) + d(p, x) = r\}$. Let r_0 be the least upper bound of the set of all $r_1 > d(o, p)$ satisfying the following properties: If $q \in E(o, p; r)$ for an $r \in (d(o, p), r_1)$, then

- (C1) there exists a minimizing geodesic segment $T(p, q)$ such that $T(p, q)$ is contained in the set $\Phi^{-1}(\widetilde{M}_p^+)$ and $\widetilde{T}(p, q) \geq U(\tilde{p}, \tilde{q})$,
- (C2) every minimizing geodesic segment $T(p, q)$ is contained in the set $\Phi^{-1}(\widetilde{M}_p^+)$ and satisfies $\widetilde{T}(p, q) \geq L(\tilde{p}, \tilde{q})$.

It follows from [2], [3] and [5] that $r_0 > d(o, p)$ (see Lemma 19). We then prove that every geodesic triangle $\triangle opq$ in M for $q \in E(o, p; r)$, $d(o, p) < r < r_0$, has a comparison triangle $\triangle \tilde{o}\tilde{p}\tilde{q}$ in \widetilde{M} satisfying (2.3) (Assertion 25). All points in $E(o, p; r_0)$ satisfy (C1) and (C2) again (Assertion 26). We see, from the assumption of our theorems, that even when $q \in E(o, p; r_0)$ with $q \notin \text{Cut}(p)$ and $\tilde{q} \in \text{Cut}(\tilde{p})$, the reference curve $\widetilde{T}_e(p, q)$ of the maximal minimizing geodesic $T_e(p, q)$ through p and q crosses $\text{Cut}(\tilde{p})$ from the far side to the near side from \tilde{o} in \widetilde{M} . This fact shows that there exists an $r' > r_0$ such that (C1) and (C2) are true for any r , $d(o, p) < r < r'$ (see Assertions 27, 28 and 29). This means that the domain bounded by $E(o, p; r_0)$ covers M .

The rest of this article is organized as follows. In §2 we state our results precisely with giving some notions we need. In §3 we give some properties of circles and ellipses in a surface of revolution. Moreover we give a sufficient condition for a point $q \in M$ not being contained in $E(p)$, and an example showing the property of ellipses which is essentially different from circles. From §4 we start studying reference curves. In §4 we give the fundamental properties of the reference curves

and the reference reverse curves. In §5 we treat the case that the reference curves from \tilde{p} do not meet $Cut(\tilde{p})$ in \widetilde{M} . In §6 we study the reference curves from \tilde{p} meeting $Cut(\tilde{p})$ from the far side to the near side from \tilde{o} . In those cases we have the good positional relation between $\widetilde{T}(p, q)$ and $T(\tilde{p}, \tilde{q})$. In §7 we give the proof of our theorems. Our assumption (2.1) ensures that $\widetilde{T}(p, q)$ crosses $Cut(\tilde{p})$ from the far side to the near side from \tilde{o} . The assumption is used only in the proofs of Assertions 26. In §8 we show some corollaries concerning the maximal perimeter and diameter as applications of our theorems. Those are the Riemannian version of Corollary 3, and we give the proof of Corollary 4.

Basic tools in Riemannian Geometry are referred to [1].

2. NOTATIONS AND STATEMENTS

Let $(\widetilde{M}, \tilde{o})$ be a surface of revolution with a geodesic polar coordinate system (r, θ) around \tilde{o} . Its metric is given by

$$ds^2 = dr^2 + f(r)^2 d\theta^2,$$

where $f(r) > 0$, $0 < r < \ell \leq \infty$, $\theta \in S^1$ and $f : [0, \ell) \rightarrow \mathbb{R}$ satisfies the Jacobi equation

$$f'' + Kf = 0, \quad f(0) = 0, \quad f'(0) = 1.$$

In addition,

$$f(\ell) = 0, \quad f'(\ell) = -1 \quad \text{if} \quad \ell < \infty.$$

The function K is called the *radial curvature function* of \widetilde{M} .

Let (M, o) be a complete Riemannian manifold with a base point at o . A *radial plane* $\Pi \subset T_p M$ at a point $p \in M$ is by definition a plane containing a vector tangent to a minimizing geodesic segment emanating from o where $T_p M$ is the tangent space of M at p . A *radial sectional curvature* $K_M(\Pi)$ is by definition a sectional curvature with respect to a radial plane Π . We say that (M, o) is *referred* to $(\widetilde{M}, \tilde{o})$ if every radial sectional curvature at $p \in M$ is bounded below by $K(d(o, p))$, namely, $K_M(\Pi) \geq K(d(o, p))$ where $d(o, p)$ is by definition the distance between o and p .

A triple of minimizing geodesic segments $T(o, p) \cup T(o, q) \cup T(p, q)$ joining points $o, p, q \in M$ is called a *geodesic triangle* and denoted by $\triangle opq$. A geodesic triangle $\triangle \tilde{o}\tilde{p}\tilde{q} \subset \widetilde{M}$ is called a *comparison triangle corresponding to $\triangle opq \subset M$* if the corresponding edges have the same lengths, namely,

$$d(o, p) = d(\tilde{o}, \tilde{p}), \quad d(o, q) = d(\tilde{o}, \tilde{q}), \quad d(p, q) = d(\tilde{p}, \tilde{q}).$$

In the reference surface of revolution \widetilde{M} every geodesic triangle $\triangle \tilde{o}\tilde{p}\tilde{q}$ bounds the region because \widetilde{M} is simply connected and the dimension of \widetilde{M} is two. The region is called a *triangle domain* and denoted by the same symbol $\triangle \tilde{o}\tilde{p}\tilde{q}$.

With respect to a point $\tilde{p} \in \widetilde{M}$, we divide \widetilde{M} into two parts as follows:

$$\widetilde{M}_{\tilde{p}}^+ = [\theta(\tilde{p}) \leq \theta \leq \theta(\tilde{p}) + \pi], \quad \widetilde{M}_{\tilde{p}}^- = [\theta(\tilde{p}) - \pi \leq \theta \leq \theta(\tilde{p})].$$

Here we set

$$[a \leq \theta \leq b] = \{\tilde{x} \in \widetilde{M} \mid a \leq \theta(\tilde{x}) \leq b\}.$$

The pair of distance functions $\tilde{x} \mapsto (d(\tilde{o}, \tilde{x}), d(\tilde{p}, \tilde{x}))$, $\tilde{x} \in \widetilde{M}$, defines a Lipschitz chart on the interiors $\text{Int}(\widetilde{M}_{\tilde{p}}^\pm)$ of $\widetilde{M}_{\tilde{p}}^\pm$ respectively (see Lemma 11 (1)). For an arbitrary fixed point $p \in M$ and $\tilde{p} \in \widetilde{M}$ we define the maps :

$$F_p : M \rightarrow \mathbb{R}^2, \quad F_p(x) = (d(o, x), d(p, x)), \quad x \in M,$$

and

$$\begin{aligned} \widetilde{F}_{\tilde{p}} : \widetilde{M}_{\tilde{p}}^+ &\rightarrow \mathbb{R}^2, & \widetilde{F}_{\tilde{p}}(\tilde{x}) &= (d(\tilde{o}, \tilde{x}), d(\tilde{p}, \tilde{x})), & \tilde{x} \in \widetilde{M}_{\tilde{p}}^+, \\ \widetilde{G}_{\tilde{p}} : \widetilde{M}_{\tilde{p}}^- &\rightarrow \mathbb{R}^2, & \widetilde{G}_{\tilde{p}}(\tilde{x}) &= (d(\tilde{o}, \tilde{x}), d(\tilde{p}, \tilde{x})), & \tilde{x} \in \widetilde{M}_{\tilde{p}}^-. \end{aligned}$$

Then, F_p , $\widetilde{F}_{\tilde{p}}$ and $\widetilde{G}_{\tilde{p}}$ are Lipschitz continuous, both $\widetilde{F}_{\tilde{p}}$ and $\widetilde{G}_{\tilde{p}}$ are injective and their inverse maps are locally Lipschitz continuous.

A unit speed minimizing geodesic segment from p to q is denoted by $T(p, q)(t)$, $0 \leq t \leq d(p, q)$, where $T(p, q)(0) = p$ and $T(p, q)(d(p, q)) = q$. Also $T(p, q)$ is identified with its image $\{T(p, q)(t) \mid 0 \leq t \leq d(p, q)\}$.

For an arbitrary fixed point $p \in M$ we set $\tilde{p} = (d(o, p), 0) \in \widetilde{M}$. If $T(p, q) \subset F_p^{-1}(\widetilde{F}_{\tilde{p}}(\widetilde{M}_{\tilde{p}}^+))$, we then define a curve $\widetilde{T}(p, q)$ in \widetilde{M} such that

$$\widetilde{T}(p, q)(t) = \widetilde{F}_{\tilde{p}}^{-1} \circ F_p(T(p, q)(t)), \quad 0 \leq t \leq d(p, q).$$

Obviously we have $\widetilde{T}(p, q)(0) = \tilde{p}$. The reference map Φ used in §1 is just $\widetilde{F}_{\tilde{p}}^{-1} \circ F_p$ which depends on the choice of $\tilde{p} \in \widetilde{M}$ corresponding to $p \in M$. Namely, if $\tilde{p}_1 = \tau_\theta(\tilde{p})$ for some rotation τ_θ of \widetilde{M} around \tilde{o} , then $\widetilde{F}_{\tilde{p}_1}^{-1} \circ F_p(T(p, q)(t)) = \tau_\theta(\widetilde{F}_{\tilde{p}}^{-1} \circ F_p(T(p, q)(t)))$.

It is convenient to use the expression $\widetilde{F}_{\tilde{p}}^{-1} \circ F_p$ for defining the reference reverse curve. Setting $\tilde{q} = \widetilde{T}(p, q)(d(p, q))$, we have the *reference reverse curve* $\widetilde{R}(p, q)$ of $T(p, q)$ which is given by

$$\widetilde{R}(p, q)(t) = \widetilde{G}_{\tilde{q}}^{-1} \circ F_q(T(p, q)(d(p, q) - t)), \quad 0 \leq t \leq d(p, q).$$

We then have $\widetilde{R}(p, q)(0) = \tilde{q}$, $\widetilde{R}(p, q)(d(p, q)) = \tilde{p}$. Both $\widetilde{T}(p, q)$ and $\widetilde{R}(p, q)$ are curves connecting \tilde{p} and \tilde{q} in \widetilde{M}_p^+ . Notice that $\widetilde{T}(p, q) \neq \widetilde{R}(p, q)$, in general, as point sets in \widetilde{M} .

A set C is said to be *parameterized by the angle coordinate* θ if $C \cap [\theta = a]$ contains at most one point where $[\theta = a] = \{\tilde{x} \in \widetilde{M} \mid \theta(\tilde{x}) = a\}$. Let two sets C_1 and C_2 be parameterized by the angle coordinate. We then define the positional relation between C_1 and C_2 by $C_1 \leq C_2$ if $r(C_1 \cap [\theta = a]) \leq r(C_2 \cap [\theta = a])$ for all $a \in \mathbb{R}$ with $C_1 \cap [\theta = a] \neq \emptyset$ and $C_2 \cap [\theta = a] \neq \emptyset$.

We say that a point \tilde{q} in \widetilde{M} is a *cut point* of \tilde{p} if any extension of a minimizing geodesic segment $T(\tilde{p}, \tilde{q})$ is not minimizing. Let $Cut(\tilde{p})$ denote the set of all cut points of $\tilde{p} \in \widetilde{M}$. It is well known that $Cut(\tilde{p})$ carries the structure of a tree in \widetilde{M} . All edges of $Cut(\tilde{p}) \cap \text{Int}(\widetilde{M}_p^\pm)$ and all non-meridian geodesics in \widetilde{M}_p^+ are parameterized by the angle coordinate.

For $r > d(o, p)$ we define an ellipsoid in M by

$$E(o, p; r) = \{x \in M \mid d(o, x) + d(p, x) = r\}$$

and the distance function to o restricted to $E(o, p; r)$ by $d_r(x) = d(o, x)$, $x \in E(o, p; r)$. Let $E_p(r)$ be the set of all points where d_r attains local maximums and set $E(p) = \cup_{r > d(o, p)} E_p(r)$. We will have $E(p) \subset Cut(p)$ (see Lemma 14).

By using these notations we will prove the following theorem which is a restatement of Theorem 1.

Theorem 5. *Assume that a complete pointed Riemannian manifold (M, o) is referred to a surface of revolution $(\widetilde{M}, \tilde{o})$. Let $p \in M$. Suppose*

$$(2.1) \quad F_p(E(p)) \cap \widetilde{F}_{\tilde{p}}(Cut(\tilde{p}) \cap \text{Int}(\widetilde{M}_p^+)) = \emptyset.$$

Then, there exists a minimizing geodesic segment $T(\tilde{p}, \tilde{q})$ in \widetilde{M}_p^+ such that

$$(2.2) \quad \widetilde{T}(p, q) \geq T(\tilde{p}, \tilde{q}) \quad \text{and} \quad \widetilde{R}(p, q) \geq T(\tilde{p}, \tilde{q}).$$

holds for every minimizing geodesic segment $T(p, q)$, $q \in M$.

Remark 6. The relation (2.2) is nothing but the Alexandrov convexity property. Namely, we have from (2.2)

$$d(o, T(p, q)(t)) \geq d(\tilde{o}, T(\tilde{p}, \tilde{q})(t)), \quad d(o, T(q, p)(t)) \geq d(\tilde{o}, T(\tilde{q}, \tilde{p})(t))$$

for all $t \in [0, d(p, q)]$ (see Lemma 18 (2)). Then the angle estimates at the corners p and q of $\triangle opq$ are obtained by the above relations (see Lemma 18 (3)).

Moreover, the angle estimate at o is obtained, also. The following theorem is the refined statement of Theorem 2.

Theorem 7. *Under the same assumptions as in Theorem 5, every geodesic triangle $\triangle opq \subset M$ admits its comparison triangle $\triangle \tilde{o}\tilde{p}\tilde{q} \subset \widetilde{M}$ such that*

$$(2.3) \quad \angle opq \geq \angle \tilde{o}\tilde{p}\tilde{q}, \quad \angle oqp \geq \angle \tilde{o}\tilde{q}\tilde{p}, \quad \angle poq \geq \angle \tilde{p}\tilde{o}\tilde{q}.$$

We emphasize that (2.3) is obtained only by the radial curvature with respect to o .

Remark 8. Under the same assumptions as in Theorem 5, if $\widetilde{T}(p, q) \cap T(\tilde{p}, \tilde{q}) \neq \{\tilde{p}, \tilde{q}\}$ for a minimizing geodesic segment $T(p, q)$ in M , then $\widetilde{T}(p, q) = T(\tilde{p}, \tilde{q})$ and a geodesic triangle $\triangle opq$ in M bounds a totally geodesic 2-dimensional submanifold which is isometric to a comparison triangle domain $\triangle \tilde{o}\tilde{p}\tilde{q}$ in \widetilde{M} corresponding to $\triangle opq$ (see Lemma 22).

Remark 9. If \widetilde{M} is the standard 2-sphere, the flat plane or the Poincaré disk, then every point $\tilde{o} \in \widetilde{M}$ is viewed as a base point of \widetilde{M} and any point $p \in M$ satisfies $Cut(\tilde{p}) \cap \text{Int}(\widetilde{M}_p^+) = \emptyset$. We say that a surface of revolution \widetilde{M} is a *von-Mangoldt surface* if its radial curvature function is monotone non-increasing. Every point on a von-Mangoldt surface of revolution \widetilde{M} satisfies $Cut(\tilde{p}) \cap \text{Int}(\widetilde{M}_p^+) = \emptyset$ (see [6]). Thus, Theorem 7 implies that (2.3) holds for every geodesic triangle $\triangle opq$. This result was first obtained in [3]. The angle estimate at the base point in a sector without cut points has been obtained by Kondo and Tanaka [5].

Remark 10. Assume that $o \in M$ is a pole of M . Namely, the exponential map $\exp_o : T_o M \rightarrow M$ at o is a diffeomorphism. Then $E(p)$ for every point $p \neq o$ is the subray from p of the meridian passing through p (see Lemma 14). We then have (2.1) for every $p \in M$, $p \neq o$, and (2.3) for every $\triangle opq$. The same fact holds for a compact Riemannian manifold M if the parameters of the first conjugate points to o along any unit speed geodesics emanating from o are constant.

Remark 10 suggests us to define a pointed Alexandrov space (M, o) with radial curvature bounded below by a function K as follows. Let $(\widetilde{M}, \tilde{o})$ be a surface of revolution with radial curvature function K . We say that an Alexandrov space (M, o) with curvature locally bounded below is a *pointed Alexandrov space with radial curvature bounded below by the function K* if the following condition is satisfied:

- (1) Every geodesic triangle $\triangle opq$ in M admits its comparison triangle $\triangle \tilde{o}\tilde{p}\tilde{q}$ in \widetilde{M} satisfying (2.2).

- (2) Conversely, for every geodesic triangle $\triangle \tilde{o}\tilde{p}\tilde{q}$ whose vertices \tilde{p} and \tilde{q} in \widetilde{M} are the reference points p and q in M , respectively, there exists a geodesic triangle $\triangle opq$ in M satisfying (2.2).

This definition makes sense because of Remark 10. In fact, if K_1 is any function less than or equal to K , then Remark 10 ensures that (M, o) is a pointed Alexandrov space with radial curvature bounded below by the function K_1 . Using this notion, we have Corollaries 3 and 4 in §1.

3. CIRCLES AND ELLIPSES

Let $S(\tilde{p}; a) = \{\tilde{x} \in \widetilde{M} \mid d(\tilde{p}, \tilde{x}) = a\}$ be the metric a -circle centered at \tilde{p} and $S(\tilde{p}, a)^\pm = S(\tilde{p}; a) \cap \widetilde{M}_p^\pm$. Like this, we often write $X^\pm := X \cap \widetilde{M}_p^\pm$ for a set $X \subset \widetilde{M}$. Let $I_{\tilde{p}, a}^\pm = \{u \in [r(\tilde{p}) - a, r(\tilde{p}) + a] \mid S(\tilde{p}; a)^\pm \cap [r = u] \neq \emptyset\}$ and $S_{\tilde{p}, a}^\pm(u) = S(\tilde{p}; a)^\pm \cap [r = u]$ for $u \in I_{\tilde{p}, a}^\pm$. We will show that $S_{\tilde{p}, a}^\pm : I_{\tilde{p}, a}^\pm \rightarrow \widetilde{M}_p^\pm$ is a union of curves. Obviously, $I_{\tilde{p}, a}^+ = I_{\tilde{p}, a}^- =: I_{\tilde{p}, a}$. In general, $S(\tilde{p}; a)$ is not necessarily connected, and then $I_{\tilde{p}, a}$ is the union of some intervals and points contained in $[r(\tilde{p}) - a, r(\tilde{p}) + a]$ as seen in Lemma 11 below. When $\ell < \infty$, we set \tilde{o}_1 to be the antipodal vertex of \tilde{o} , namely $r(\tilde{o}_1) = \ell$. Obviously, $S(\tilde{o}; a) = S(\tilde{o}_1; \ell - a) = [r = a]$, $0 \leq a \leq \ell$.

Lemma 11. *Let $\tilde{p} \in \widetilde{M}$, $\tilde{p} \neq \tilde{o}, \tilde{o}_1$. The metric circles in \widetilde{M} satisfy the following properties.*

- (1) *If an r_1 -parallel circle $c = [r = r_1] \subset \widetilde{M}_p^+$ is parameterized as $c(\varphi) = [r = r_1] \cap [\theta = \varphi]$ for any φ , then $d(\tilde{p}, c(\varphi))$ is strictly increasing in $\varphi \in [\theta(\tilde{p}), \theta(\tilde{p}) + \pi]$.*
- (2) *Each of $S_{\tilde{p}, a}^+(u)$ and $S_{\tilde{p}, a}^-(u)$ consists of only one point for any $u \in I_{\tilde{p}, a}$. In particular, each of $S_{\tilde{p}, a}^+$ and $S_{\tilde{p}, a}^-$ is the union of curves in \widetilde{M} with parameter $u \in I_{\tilde{p}, a}$.*
- (3) *Let $\tilde{q} \in \widetilde{M}_p^+$ and let $s > 0$ and $t > 0$ satisfy $s + t = d(\tilde{p}, \tilde{q})$. If $\tilde{z} \in S(\tilde{p}, s)^+ \cap S(\tilde{q}, t)^-$, then there exists the unique minimizing geodesic segment $T(\tilde{p}, \tilde{q})$ passing through \tilde{z} .*

Proof. Since all meridians $[\theta = \varphi]$ are geodesics which intersect the parallel circles $[r = r_1]$ orthogonally and hence the minimizing geodesic segments from \tilde{p} intersect the parallel circle $c = [r = r_1]$ with the angles less than $\pi/2$, we have (1) from the first variation formula.

This fact (1) implies that each of $S_{\tilde{p}, a}^+(u)$ and $S_{\tilde{p}, a}^-(u)$ consists of at most one point for any $u \in [r(\tilde{p}) - a, r(\tilde{p}) + a]$. This proves (2).

Then (3) follows from the fact

$$d(\tilde{p}, \tilde{z}) + d(\tilde{z}, \tilde{q}) = s + t = d(\tilde{p}, \tilde{q}).$$

This completes the proof. \square

Let $T(p, q)(0)$ denote the tangent vector of the curve $T(p, q)(t)$ at $t = 0$. We define a map $g : [0, \pi] \rightarrow S(\tilde{p}; a)^+ \cup \{\phi\}$ as follows: If $\omega \in [0, \pi]$ is the angle of $T(\tilde{p}, \tilde{o})(0)$ with $T(\tilde{p}, \tilde{q})(0)$ for some point $\tilde{q} \in S(\tilde{p}; a)$, then $g(\omega) = \tilde{q}$. If ω satisfies $\omega_1 \leq \omega \leq \omega_2$ for some ω_1 and ω_2 with $g(\omega_1) = g(\omega_2)$, then $g(\omega) = g(\omega_1)$. Otherwise, $g(\omega) = \phi$ where ϕ is the dummy. The connected components of $[0, \pi] \setminus g^{-1}(\phi)$ corresponds to those of $I_{\tilde{p}, a}$. If $\bar{g} : [0, \pi] \setminus g^{-1}(\phi) \rightarrow I_{\tilde{p}, a}$ is the map given by $g(\omega) = S_{\tilde{p}, a}^+(\bar{g}(\omega))$, then \bar{g} is monotone nondecreasing in each connected component.

Let $B(\tilde{o}, \tilde{p}; a) \subset \widetilde{M}$ for $a > d(\tilde{o}, \tilde{p})$ be the domain given by $B(\tilde{o}, \tilde{p}; a) = \{\tilde{q} \mid d(\tilde{o}, \tilde{q}) + d(\tilde{p}, \tilde{q}) \leq a\}$. When $\ell < \infty$, the function $d(\tilde{p}, \tilde{q}) + d(\tilde{o}, \tilde{q})$, $\tilde{q} \in \widetilde{M}$, attains the maximum $2\ell - d(\tilde{p}, \tilde{o})$ at \tilde{o}_1 . We therefore have $B(\tilde{o}, \tilde{p}; a) \supset \widetilde{M}$ for every $a \geq 2\ell - d(\tilde{p}, \tilde{o})$. If $a < 2\ell - d(\tilde{p}, \tilde{o})$, then $\tilde{o}_1 \notin B(\tilde{o}, \tilde{p}; a)$.

Lemma 12. *Let $\tilde{p} \neq \tilde{o}, \tilde{o}_1$. The ellipses $E(\tilde{o}, \tilde{p}; a)$, $d(\tilde{o}, \tilde{p}) < a < 2\ell - d(\tilde{o}, \tilde{p})$, in \widetilde{M} has the following properties.*

- (1) $B(\tilde{o}, \tilde{p}; a)$ is star-shaped around \tilde{p} and \tilde{o} . Namely, $T(\tilde{p}, \tilde{q}) \subset B(\tilde{o}, \tilde{p}; a)$ and $T(\tilde{o}, \tilde{q}) \subset B(\tilde{o}, \tilde{p}; a)$ for any $\tilde{q} \in B(\tilde{o}, \tilde{p}; a)$. If $\tilde{q} \in E(\tilde{o}, \tilde{p}; a)$, then $T(\tilde{p}, \tilde{q}) \cap E(\tilde{o}, \tilde{p}; a) = \{\tilde{q}\}$ and $T(\tilde{o}, \tilde{q}) \cap E(\tilde{o}, \tilde{p}; a) = \{\tilde{q}\}$. Furthermore, $\angle \tilde{p}\tilde{q}\tilde{o} \neq \pi$.
- (2) The intersection $E(\tilde{o}, \tilde{p}; a) \cap [\theta = \varphi]$ is a single point for all $\varphi \in [\theta(\tilde{p}) - \pi, \theta(\tilde{p}) + \pi]$. If $(r(\varphi), \varphi) = E(\tilde{o}, \tilde{p}; a) \cap [\theta = \varphi]$, then $r(\theta(\tilde{p}) - \varphi) = r(\theta(\tilde{p}) + \varphi)$ for $\varphi \in [0, \pi]$. Moreover, $r(\varphi)$ is monotone increasing for $\varphi \in [\theta(\tilde{p}) - \pi, \theta(\tilde{p})]$ and monotone decreasing for $\varphi \in [\theta(\tilde{p}), \theta(\tilde{p}) + \pi]$. In particular, $r(\theta(\tilde{p})) = (a + d(\tilde{o}, \tilde{p}))/2$ is the maximum and $r(\theta(\tilde{p}) \pm \pi)$ is the minimum.
- (3) If $e^\pm(u) = E(\tilde{o}, \tilde{p}; a) \cap \widetilde{M}_{\tilde{p}}^\pm \cap [r = u]$ for $u \in [r(\theta(\tilde{p}) + \pi), r(\theta(\tilde{p}))]$, then the function $d(\tilde{p}, e^\pm(u))$ is monotone decreasing in $u \in [r(\theta(\tilde{p}) + \pi), r(\theta(\tilde{p}))]$.
- (4) Set $\tilde{q}_1 = (r(\theta(\tilde{p}) \pm \pi), \theta(\tilde{p}) \pm \pi)$. Let $b \in (a - r(\theta(\tilde{p})), d(\tilde{p}, \tilde{q}_1))$. Then, $S(\tilde{p}, b)$ crosses $E(\tilde{o}, \tilde{p}; a)$ once in each of $\widetilde{M}_{\tilde{p}}^+$ and $\widetilde{M}_{\tilde{p}}^-$. If $b = (a - d(\tilde{o}, \tilde{p}))/2$, then $S(\tilde{p}, b) \subset B(\tilde{o}, \tilde{p}; a)$ and $S(\tilde{p}, b) \cap E(\tilde{o}, \tilde{p}; a) = \{(r(\theta(\tilde{p})), \theta(\tilde{p}))\}$. If $b = d(\tilde{o}, \tilde{q}_1)$, then $S(\tilde{o}, b) \subset B(\tilde{o}, \tilde{p}; a)$ and $S(\tilde{o}, b) \cap E(\tilde{o}, \tilde{p}; a) = \{\tilde{q}_1\}$.

To be seen in Example 15, the third statement of (1) is not true, in general, for a complete pointed Riemannian manifold (M, o) . Namely, $T(p, q) \cup T(q, o)$ may be a geodesic segment from p to o via q with $q \in E(o, p; a)$.

Proof. Let $\tilde{q} \in B(\tilde{o}, \tilde{p}; a)$ and let $\tilde{q}' \in T(\tilde{p}, \tilde{q})$. We then have

$$\begin{aligned} d(\tilde{o}, \tilde{q}') + d(\tilde{p}, \tilde{q}') &= d(\tilde{o}, \tilde{q}') + d(\tilde{p}, \tilde{q}) - d(\tilde{q}, \tilde{q}') \\ &\leq d(\tilde{o}, \tilde{q}) + d(\tilde{p}, \tilde{q}) \leq a. \end{aligned}$$

This means that $\tilde{q}' \in B(\tilde{o}, \tilde{p}; a)$, and, hence, $T(\tilde{p}, \tilde{q}) \subset B(\tilde{o}, \tilde{p}; a)$. In the same way we have $T(\tilde{o}, \tilde{q}) \subset B(\tilde{o}, \tilde{p}; a)$. These prove the first part of (1).

Suppose there exists a point $\tilde{q}' \in E(o, p; a) \cap T(\tilde{p}, \tilde{q}) \setminus \{\tilde{q}\}$. Then the above inequality shows that $d(\tilde{o}, \tilde{q}') = d(\tilde{o}, \tilde{q}) + d(\tilde{q}, \tilde{q}')$. From this we have $T(\tilde{p}, \tilde{q}) \cap T(\tilde{o}, \tilde{q}) \supset T(\tilde{q}', \tilde{q})$, meaning that $T(\tilde{p}, \tilde{q}) \cup T(\tilde{q}, \tilde{o})$ is a geodesic connecting \tilde{p} and \tilde{o} which is different from the meridian passing through \tilde{p} , a contradiction. This proves the second part of (1).

The third part of (1) is obvious. In fact, if $\angle \tilde{p}\tilde{q}\tilde{o} = \pi$, then $d(\tilde{p}, \tilde{q}) + d(\tilde{q}, \tilde{o}) + d(\tilde{o}, \tilde{p}) = 2\ell$, meaning that $a = 2\ell - d(\tilde{o}, \tilde{p})$, a contradiction.

Notice that the function $f(r) = d(\tilde{p}, (r, \varphi)) + r$ is monotone increasing in $r \in (0, \ell)$ with $f(r) > r(\tilde{p})$ because of the first variation formula. Since $\sup\{f(r) \mid r \in (0, \ell)\} = 2\ell - d(\tilde{o}, \tilde{p})$, we have the first part of (2).

Let φ_1 and φ_2 be such that $\theta(\tilde{p}) \leq \varphi_1 < \varphi_2 \leq \theta(\tilde{p}) + \pi$. If $r(\varphi_1) = r(\varphi_2)$, then $d(\tilde{p}, (r(\varphi_1), \varphi_1)) = d(\tilde{p}, (r(\varphi_2), \varphi_2))$, which contradicts Lemma 11 (1). By the triangle inequality,

$$\begin{aligned} r(\varphi) &\leq d(\tilde{o}, \tilde{p}) + d(\tilde{p}, (r(\varphi), \varphi)) \\ &= d(\tilde{o}, \tilde{p}) + a - r(\varphi), \end{aligned}$$

and, hence, we have $r(\varphi) \leq (a + d(\tilde{o}, \tilde{p}))/2$ where the equality holds if and only if $\varphi = \theta(\tilde{p})$. These imply the other parts of (2) and (3).

If $b \in (a - r(\theta(\tilde{p})), a - r(\theta(\tilde{p}) + \pi))$, then each curve $(r(\varphi), \varphi)$ for $\varphi \in [\theta(\tilde{p}) - \pi, \theta(\tilde{p})]$ and $[\theta(\tilde{p}), \theta(\tilde{p}) + \pi]$ moves from the outside of $S(\tilde{p}, b)$ to its inside and from its inside to its outside, respectively. The property (2) of this lemma implies that the crossing point is unique in each curve, which proves the first part of (4).

In order to prove the second part of (4), let $b = (a - d(\tilde{o}, \tilde{p}))/2$ and $\tilde{q} \in S(\tilde{p}, b)$. We then have

$$d(\tilde{o}, \tilde{q}) + d(\tilde{p}, \tilde{q}) \leq d(\tilde{o}, \tilde{p}) + 2d(\tilde{p}, \tilde{q}) = a$$

and equality holding if and only if $d(\tilde{o}, \tilde{p}) + d(\tilde{p}, \tilde{q}) = d(\tilde{o}, \tilde{q})$. This means that $S(\tilde{p}, b) \subset B(\tilde{o}, \tilde{p}; a)$ and $S(\tilde{p}, b) \cap E(\tilde{o}, \tilde{p}; a) = \{(r(\theta(\tilde{p})), \theta(\tilde{p}))\}$.

Let $b = d(\tilde{o}, \tilde{q}_1)$ and $\tilde{q} \in S(\tilde{o}, b)$, namely $d(\tilde{o}, \tilde{q}) = d(\tilde{o}, \tilde{q}_1)$. We then have, from Lemma 11 (1),

$$d(\tilde{o}, \tilde{q}) + d(\tilde{p}, \tilde{q}) \leq d(\tilde{o}, \tilde{q}_1) + d(\tilde{p}, \tilde{q}_1) = a,$$

and equality holding if and only if $\tilde{q} = \tilde{q}_1$. This proves the third part of (4). \square

Let $\tilde{p} \neq \tilde{o}$, \tilde{o}_1 and $d(\tilde{o}, \tilde{p}) < a < 2\ell - d(\tilde{o}, \tilde{p})$. The reference curves will be made in $\widetilde{M}_{\tilde{p}}^+$, so we work in $\widetilde{M}_{\tilde{p}}^+$. From Lemma 12 the set $\Omega(\tilde{o}, \tilde{p}; a)^+$ of all minimizing geodesic segments $T(\tilde{p}, \tilde{q})$ from \tilde{p} to points $\tilde{q} \in E(\tilde{o}, \tilde{p}; a)^+$ is a totally ordered set with respect to the binary relation \leq in the set of curves in $\widetilde{M}_{\tilde{p}}^+$. The minimizing geodesic segments $T(\tilde{p}, \tilde{q})$, $\tilde{q} \in E(\tilde{o}, \tilde{p}; a)^+$, divide $B(\tilde{o}, \tilde{p}; a)^+$ into two domains. Let $U(\tilde{p}, \tilde{q})$ denote the greatest minimizing geodesic segment connecting \tilde{p} and \tilde{q} and $L(\tilde{p}, \tilde{q})$ the least one. Namely $L(\tilde{p}, \tilde{q}) \leq T(\tilde{p}, \tilde{q}) \leq U(\tilde{p}, \tilde{q})$ for every minimizing geodesic segment $T(\tilde{p}, \tilde{q})$. If $\tilde{q} \notin \text{Cut}(\tilde{p})$, then $U(\tilde{p}, \tilde{q}) = L(\tilde{p}, \tilde{q})$. If $U(\tilde{p}, \tilde{q}) \neq L(\tilde{p}, \tilde{q})$ for a point $\tilde{q} \in \text{Cut}(\tilde{p})^+$, then $B(\tilde{o}, \tilde{p}; a)^+$ is divided into three domains B_1 , B_0 and B_2 . Here B_1 is the domain bounded by the meridian $[\theta = \theta(\tilde{p})]$, $E(\tilde{o}, \tilde{p}; a)^+$ and $U(\tilde{p}, \tilde{q})$, B_0 is the biangle domain bounded by $U(\tilde{p}, \tilde{q})$ and $L(\tilde{p}, \tilde{q})$, B_2 is the domain bounded by the meridians $[\theta = \theta(\tilde{p})] \cup [\theta = \theta(\tilde{p}) + \pi]$, $E(\tilde{o}, \tilde{p}; a)^+$ and $L(\tilde{p}, \tilde{q})$.

Lemma 13. *Let $\tilde{p} \neq \tilde{o}$, \tilde{o}_1 and $d(\tilde{o}, \tilde{p}) < a < 2\ell - d(\tilde{o}, \tilde{p})$. Let $\tilde{q} \in E(\tilde{o}, \tilde{p}; a)^+$. Let \tilde{q}' be a sequence of points in $E(\tilde{o}, \tilde{p}; a)^+$ such that $r(\tilde{q}') > r(\tilde{q})$ (resp., $r(\tilde{q}') < r(\tilde{q})$) and it converges to \tilde{q} . Then the sequence of segments $T(\tilde{p}, \tilde{q}')$ converges to $U(\tilde{p}, \tilde{q})$ (resp., $L(\tilde{p}, \tilde{q})$).*

Proof. A subsequence of the sequence $T(\tilde{p}, \tilde{q}')$ converges to a minimizing geodesic segment $T(\tilde{p}, \tilde{q})$. Since B_1 and B_2 are star-shaped around \tilde{p} , it follows that $T(\tilde{p}, \tilde{q}')$ is contained in either B_1 or B_2 , depending on $r(\tilde{q}') > r(\tilde{q})$ or $r(\tilde{q}') < r(\tilde{q})$. From the definition of $U(\tilde{p}, \tilde{q})$ and $L(\tilde{p}, \tilde{q})$, it follows that $T(\tilde{p}, \tilde{q})$ is one of $U(\tilde{p}, \tilde{q})$ and $L(\tilde{p}, \tilde{q})$. This shows that the sequence of minimizing geodesic segments $T(\tilde{p}, \tilde{q}')$ converges to either $U(\tilde{p}, \tilde{q})$ or $L(\tilde{p}, \tilde{q})$. \square

We now discuss the property of ellipses in M , which includes new ideas and play an important role. The following lemma gives a sufficient condition for $q \notin E_p(r)$, namely q is not a local maximum point of the distance function d_r to o restricted to $E(o, p; r)$.

Lemma 14. *Let $q \in E(o, p; r) \subset M$. If there exists a point u such that $d(p, u) + d(o, u) > r$ and $d(p, u) - d(p, q) < d(o, u) - d(o, q)$, then there exists a point $q' \in E(o, p; r)$ such that $d(o, q') > d(o, q)$. In particular, if $q \notin \text{Cut}(o)$ and $p \notin T(o, q)$, then q is not a local maximum point of d_r on $E(o, p; r)$.*

We observe that the assumption $d(p, u) + d(o, q) < d(p, q) + d(o, u)$ means that $d(o, T(q, u)(t))$ increase further than $d(p, T(q, u)(t))$ for $t \in [0, d(q, u)]$.

Proof. We first prove that the set $E(o, p; r) \cap T(p, u)$ consists of a single point, say q' . Suppose there exists a point $q'' \in E(o, p; r) \cap T(p, u)$ with $q'' \neq q'$. Assume without loss of generality that p, q', q'', u are in this order in $T(p, u)$. Since

$$\begin{aligned} d(p, q') + d(o, q') &= r \\ &= d(p, q'') + d(o, q'') \\ &= d(p, q') + d(q', q'') + d(o, q''), \end{aligned}$$

we have $d(o, q') = d(q', q'') + d(o, q'')$. This means that the minimizing geodesic segment $T(q', q'')$ is contained in both segments $T(o, q')$ and $T(p, u)$. In particular, $u \in T(o, q')$. This is a contradiction, because

$$\begin{aligned} r < d(p, u) + d(o, u) &= d(p, q'') + d(q'', u) + d(o, u) \\ &= d(p, q'') + d(o, q'') = r. \end{aligned}$$

We should note that $q' \neq q$. In fact, if $q' = q$, we then have

$$\begin{aligned} d(q', u) &= d(p, u) - d(p, q') \\ &= d(p, u) - d(p, q) \\ &< d(o, u) - d(o, q') \leq d(q', u), \end{aligned}$$

a contradiction.

We next prove that $d(o, q') > d(o, q)$. If $d(p, q') < d(p, q)$, we have

$$\begin{aligned} d(o, q') &= r - d(p, q') \\ &> r - d(p, q) \\ &= d(o, q). \end{aligned}$$

Thus we suppose $d(p, q') \geq d(p, q)$. We then have

$$\begin{aligned} d(o, q') &\geq d(o, u) - d(u, q') \\ &= d(o, u) - d(p, u) + d(p, q') \\ &\geq d(o, u) - d(p, u) + d(p, q) \\ &> d(o, u) - d(o, u) + d(o, q) \\ &= d(o, q). \end{aligned}$$

This completes the proof of the first part of the lemma.

Assume that $q \notin \text{Cut}(o)$ and $p \notin T(o, q)$. Let u be a point such that $q \in T(o, u)$ with $u \neq q$. We have

$$\begin{aligned} d(p, u) + d(o, u) &= d(p, u) + d(u, q) + d(q, o) \\ &\geq d(p, q) + d(o, q) = r, \end{aligned}$$

where the equality holds if and only if $d(p, q) = d(p, u) + d(u, q)$. By the triangle inequality, we have

$$d(p, u) - d(p, q) \leq d(q, u) = d(o, u) - d(o, q),$$

where the equality holds if and only if $d(p, u) = d(p, q) + d(q, u)$.

In order to prove that q is not a local maximum point of d_r on $E(o, p; r)$, we have to discuss the equality cases. Suppose first that $d(p, u) + d(u, q) = d(p, q)$. Then, $T(u, q) \subset T(p, q) \cap T(o, u)$, which means that $T(u, q) \subset E(o, p; r)$. Namely, $T(p, q) \cup T(u, o)$ is a geodesic in M connecting p and o which is not minimizing such that the subsegment from u to q is contained in $E(o, p; r)$. Such a geodesic will be seen in Example 15. Since every point $q' \in T(q, u) \setminus \{q\}$ satisfies that $d(o, q') > d(o, q)$, the point q is not a local maximal point of the function d_r .

We next suppose $d(p, u) = d(p, q) + d(q, u)$. Then, $T(q, u) \subset T(p, u) \cap T(o, u)$. Since $p \notin T(o, q)$, we have $o \in T(p, q) \subset T(p, u)$. Since $q \notin \text{Cut}(p)$, the set $S(p, d(p, q)) = \{u' \mid d(p, u') = d(p, q)\}$ contains a set U around q which is homeomorphic to a disk with dimension $\dim M - 1$ and any point $u' \in U$ with $u' \neq q$ satisfies $d(o, u') > d(o, q)$. In fact, $d(o, u') > |d(p, u') - d(p, o)| = |d(p, q) - d(p, o)| = d(o, q)$. Thus, we can find a point u' near q satisfying the assumption in the first part of the lemma, namely $d(p, u') + d(o, u') > r$ and $d(p, u') - d(p, q) < d(o, u') - d(o, q)$. From these arguments we may assume without loss of generality that there exists a point u' near q satisfying the assumption in the first part.

It remains to find a point $q'' \in E(o, p; r)$ near q such that $d(o, q'') > d(o, q)$. Let u' be a sequence of points satisfying the assumption in the first part of the lemma and converging to q . Let $q'(u') = E(o, p; r) \cap T(p, u')$ which satisfies $d(o, q'(u')) > d(o, q)$. The sequence of minimizing geodesic segments $T(q'(u'), u')$ converges to the point q or it contains a subsequence converging to a minimizing geodesic segment $T(q', q)$ contained in $E(o, p; r)$ as u' goes to q . When the first case occurs, the existence of $q'(u')$ shows that q is not a local maximum point of d_r . Suppose the second case happens. If $q'' \in T(q', q)$, we then have

$$\begin{aligned} d(o, q'') &= r - d(p, q'') \\ &= r - (d(p, q) - d(q'', q)) \\ &> r - d(p, q) = d(o, q). \end{aligned}$$

This implies that q is not a local maximum point of d_r on $E(o, p; r)$. This completes the proof. \square

The following example is helpful to understand what happens on ellipses as being larger. It should be noted that there exists a point $q \in E(o, p; r)$ which cannot be an accumulation point of interior points of $B(o, p; r)$.

Example 15. We study how ellipses change in a flat cylinder as being larger. Let $M = \{(x, y, z) \in \mathbb{E}^3 \mid x^2 + z^2 = 1\}$. Let $o = (1, 0, 0)$ and $p = (0, 2, -1)$. Then $Cut(o) = \{(-1, y, 0) \mid y \in \mathbb{R}\}$ and $Cut(p) = \{(0, y, 1) \mid y \in \mathbb{R}\}$. We identify M with \mathbb{E}^2/Γ where $\mathbb{E}^2 = \{(x, y) \mid x, y \in \mathbb{R}\}$ and Γ is the isometry group generated by a translation τ such that $\tau((x, y)) = (x, y + 2\pi)$. The universal covering space $\pi : \mathbb{E}^2 \rightarrow M$ is given by $\pi((y, \theta)) = (\cos \theta, y, \sin \theta)$. The tangent plane M_o is identified with \mathbb{E}^2 also. Then $\tilde{C}(o) = \{(x, \pm\pi) \mid x \in \mathbb{R}\}$ is the tangent cut locus of o and $U = \{(x, y) \mid x \in \mathbb{R}, -\pi \leq y \leq \pi\}$ is the lift of the normal coordinate neighborhood of o , namely $\exp_o : U \setminus \tilde{C}(o) \rightarrow M \setminus Cut(o)$ is a diffeomorphism. If $\varphi = \exp_o|_U$, then $\varphi^{-1}(o) = (0, 0) =: o_0$ and $\varphi^{-1}(p) = (2, -\pi/2) =: p_0$ by this identification. Set $p_1 = (2, 3\pi/2)$. Further, $\varphi^{-1}(Cut(p)) = \{(x, \pi/2) \mid x \in \mathbb{R}\}$. Let $E(o, p; r) = \{w \mid F(w) := d(o, w) + d(p, w) = r\}$ and $B(o, p; r) = \{w \mid F(w) \leq r\}$ for each $r > d(o, p)$.

Set $r_0 = \min\{F(w) \mid w \in Cut(o)\}$, $\{a\} = E(o_0, p_0; r_0) \cap T(o_0, p_1)$, $\{q_1\} = \tilde{C}(o) \cap T(o_0, p_1)$. Let ∂X denote the boundary of a subset X . Then $\varphi^{-1}(E(o, p; r))$ changes for r as follows.

- (1) $\varphi^{-1}(E(o, p; r)) = E(o_0, p_0; r)$ if r satisfies $d(o, p) < r < r_0$.
- (2) $\varphi^{-1}(E(o, p; r_0)) = E(o_0, p_0; r_0) \cup T(a, q_1)$.
- (3) $\varphi^{-1}(E(o, p; r)) = \partial(B(o_0, p_0; r) \cup B(o_0, p_1; r)) \cap U$ if r satisfies $r > r_0$.

Let $q_2 = \tau^{-1}(q_1)$. If $q \in Cut(o)$ satisfies $F(q) = \min F|_{Cut(o)}$, then $\varphi^{-1}(q) = \{q_1, q_2\}$. Moreover, $\varphi(T(o_0, q_1) \cup T(q_2, p_0))$ is a geodesic connecting o and p in M . The geodesic reflecting against $Cut(o)$ at q in M is identified with $\varphi(T(o_0, q_2) \cup T(q_2, p_0))$.

It should be remarked that any sequence of points q'_j such that $q'_j \in E(o, p; r_j)$ for $r_j < r_0$ with $r_j \rightarrow r_0$ cannot converge to any point in $\varphi(T(a, q_1) \setminus \{a, q_1\}) \subset E(o, p; r_0)$. Thus, we notice that there exists a geodesic triangle $\triangle opq$ with $q \in E(o, p; r_0)$ such that it admits no sequence of geodesic triangles $\triangle opq_j$ with $q_j \in E(o, p; r_j)$, $r_j < r_0$, converging to itself.

4. REFERENCE CURVES

Let $(\widetilde{M}, \tilde{o})$ be a surface of revolution with vertex \tilde{o} . Throughout this section, we do not assume that (M, o) is referred to $(\widetilde{M}, \tilde{o})$ and

$F_p(E(p)) \cap \tilde{F}_{\tilde{p}}(\text{Cut}(\tilde{p}) \cap \text{Int}(\tilde{M}_{\tilde{p}}^+)) = \emptyset$. However, we assume that every minimizing geodesic segment $T(p, q)$ in consideration is contained in $F_p^{-1}(\tilde{F}_{\tilde{p}}(\tilde{M}_{\tilde{p}}^+))$. Therefore, $\tilde{T}(p, q)(t)$ is defined for all $t \in [0, d(p, q)]$.

Lemma 16. *Let $\tilde{q}(t) = \tilde{T}(p, q)(t)$, $0 \leq t \leq d(p, q)$. Let I denote the set of all parameters $t \in [0, d(p, q)]$ such that $\tilde{q}(t) \in \text{Int}(\tilde{M}_{\tilde{p}}^+)$. The curves $\tilde{T}(p, q)$ and $\tilde{R}(p, q)$ satisfy the following properties.*

- (1) *I is an interval. $\theta(\tilde{q}(t))$ is monotone increasing for $t \in I$. More precisely, if $t_0 := \max\{t \in [0, d(p, q)] \mid \theta(\tilde{q}(t)) = \theta(\tilde{p})\} > 0$, we then have two possibilities: If $\tilde{q}(t_0) \in T(\tilde{p}, \tilde{o})$, then $T(p, q)$ is contained in the maximal minimizing geodesic segment $T_e(p, o)$ from p through o and $\tilde{T}(p, q)$ is contained in the union of meridians $[\theta = \theta(\tilde{p})] \cup [\theta = \theta(\tilde{p}) + \pi]$. If $\tilde{p} \in T(\tilde{o}, \tilde{q}(t_0))$, then $T(p, q(t_0)) = T(o, q(t_0)) \cap T(p, q)$ and $\tilde{T}(p, q)([0, t_0])$ is contained in the meridian through \tilde{p} . In addition, if $t'_0 := \min\{t \in [0, d(p, q)] \mid \theta(\tilde{q}(t)) = \theta(\tilde{p}) + \pi\} < d(p, q)$, we then have the similar results as above by using q and \tilde{q} instead of p and \tilde{p} .*
- (2) *$\tilde{R}(p, q)(t)$ is defined for all $t \in [0, d(p, q)]$.*
- (3) *$d(\tilde{p}, \tilde{T}(p, q)(t)) = t$, $d(\tilde{q}, \tilde{R}(p, q)(t)) = t$, $0 \leq t \leq d(p, q)$.*
- (4) *$d(\tilde{p}, \tilde{T}(p, q)(t)) + d(\tilde{q}, \tilde{R}(p, q)(d(p, q) - t)) = d(p, q) = d(\tilde{p}, \tilde{q})$.*
- (5) *$r(\tilde{T}(p, q)(t)) = r(\tilde{R}(p, q)(d(p, q) - t))$.*
- (6) *$\tilde{T}(p, q) \supset \tilde{T}(p, q')$ and $\tilde{R}(p, q) \supset \tilde{R}(q', q)$ for any point $q' \in T(p, q)$.*

Proof. Let $q(t) = T(p, q)(t)$. We first prove that if there exist two parameters t_1 and t_2 such that $t_1 < t_2$ and $\theta(\tilde{q}(t_1)) = \theta(\tilde{q}(t_2))$ or $\tilde{q}(t_1) = \tilde{o}$ (or $\tilde{q}(t_1) = \tilde{o}_1$ if $\ell < \infty$), then $d(\tilde{p}, \tilde{q}(t_2)) = d(\tilde{p}, \tilde{q}(t_1)) + d(\tilde{q}(t_1), \tilde{q}(t_2))$, namely $\tilde{q}(t_1) \in T(\tilde{p}, \tilde{q}(t_2))$. In fact, since $\theta(\tilde{q}(t_1)) = \theta(\tilde{q}(t_2))$ implies that $|r(\tilde{q}(t_2)) - r(\tilde{q}(t_1))| = d(\tilde{q}(t_1), \tilde{q}(t_2))$, we have

$$\begin{aligned}
d(\tilde{p}, \tilde{q}(t_2)) - d(\tilde{p}, \tilde{q}(t_1)) &= d(p, q(t_2)) - d(p, q(t_1)) \\
&= d(q(t_1), q(t_2)) \\
&\geq |d(o, q(t_2)) - d(o, q(t_1))| \\
&= |r(\tilde{q}(t_2)) - r(\tilde{q}(t_1))| \\
&= d(\tilde{q}(t_1), \tilde{q}(t_2)) \\
&\geq d(\tilde{p}, \tilde{q}(t_2)) - d(\tilde{p}, \tilde{q}(t_1)),
\end{aligned}$$

meaning that $\tilde{q}(t_1) \in T(\tilde{p}, \tilde{q}(t_2))$. Thus, $T(\tilde{p}, \tilde{q}(t_2))$ is contained in the union of the meridians $[\theta = \theta(\tilde{p})] \cup [\theta = \theta(\tilde{p}) + \pi]$. Therefore, $\theta(\tilde{q}(t))$ is monotone increasing in the interval $I \subset [0, d(p, q)]$ such that $\tilde{q}(I) \subset \text{Int}(\tilde{M}_{\tilde{p}}^+)$.

Suppose $t_0 > 0$. We have to treat two cases; $\tilde{q}(t_0) \in T(\tilde{p}, \tilde{o})$ and $\tilde{p} \in T(\tilde{o}, \tilde{q}(t_0))$. In the first case, we have

$$\begin{aligned} d(p, q(t_0)) + d(q(t_0), o) &= d(\tilde{p}, \tilde{q}(t_0)) + d(\tilde{q}(t_0), \tilde{o}) \\ &= d(\tilde{p}, \tilde{o}) \\ &= d(p, o), \end{aligned}$$

meaning that $T(p, q(t_0)) \subset T(p, o)$. Therefore, $T(p, q) \subset T_e(p, o)$, and therefore $\tilde{T}(\tilde{p}, \tilde{q})$ is contained in the union of the meridian through \tilde{p} and the meridian opposite to \tilde{p} . In the second case, we have

$$\begin{aligned} d(o, p) + d(p, q(t_0)) &= d(\tilde{o}, \tilde{p}) + d(\tilde{p}, \tilde{q}(t_0)) \\ &= d(\tilde{o}, \tilde{q}(t_0)) \\ &= d(o, q(t_0)), \end{aligned}$$

meaning that $T(p, q(t_0)) \subset T(o, q(t_0))$. Therefore, $\tilde{T}(p, q)([0, t_0])$ is contained in the meridian through \tilde{p} .

It remains to prove that $\theta(\tilde{q}(t)) = \theta(\tilde{p}) + \pi$ for $t > t'_0$ in case of $\theta(\tilde{q}(t'_0)) = \theta(\tilde{p}) + \pi$. Suppose that there exists a parameter $t \in (t'_0, d(p, q))$ such that $\theta(\tilde{p}) < \theta(\tilde{q}(t)) < \theta(\tilde{p}) + \pi$. We then find a parameter $t_3 \in (0, t'_0)$ such that $\theta(\tilde{q}(t_3)) = \theta(\tilde{q}(t))$ or $\tilde{q}(t_3) = \tilde{o}$ (or $\tilde{q}(t_3) = \tilde{o}_1$). By the same argument as above, we have a contradiction. In particular, $\theta(q(t'_0)) = \theta(\tilde{q}(d(p, q))) = \theta(\tilde{p}) + \pi$. From the argument above, it follows that $T(\tilde{p}, \tilde{q})$ is contained in the union of the meridians $[\theta = \theta(\tilde{p})] \cup [\theta = \theta(\tilde{p}) + \pi]$. We have proved (1).

In order to prove (2) we suppose that $\theta(\tilde{p}(t_0)) = \theta(\tilde{p})$ for some $t_0 \in [0, d(p, q))$ where $\tilde{p}(t) = \tilde{R}(p, q)(t)$, $0 \leq t \leq d(p, q)$. Then, we have

$$\begin{aligned} d(p, q) &= t_0 + d(p, q) - t_0 \\ &= d(\tilde{q}, \tilde{p}(t_0)) + d(\tilde{p}, \tilde{q}(d(p, q) - t_0)) \\ &\geq d(\tilde{q}, \tilde{p}(t_0)) + d(\tilde{p}, \tilde{p}(t_0)) \\ &\geq d(\tilde{p}, \tilde{q}) \\ &= d(p, q), \end{aligned}$$

since $r(\tilde{p}(t_0)) = r(\tilde{q}(d(p, q) - t_0)) = d(o, q(d(p, q) - t_0))$ and $\tilde{p}(t_0)$ lies in the meridian through \tilde{p} and because of Lemma 11 (1). As before, $\tilde{R}([t_0, d(p, q)])$ lies on the meridian through \tilde{p} . This shows (2).

Since

$$\begin{aligned} &(d(\tilde{o}, \tilde{T}(p, q)(t)), d(\tilde{p}, \tilde{T}(p, q)(t))) \\ &= \tilde{F}_{\tilde{p}}(\tilde{T}(p, q)(t)) \\ &= F_p(T(p, q)(t)) \\ &= (d(o, T(p, q)(t)), d(p, T(p, q)(t))), \end{aligned}$$

we have

$$r(\tilde{T}(p, q)(t)) = d(o, T(p, q)(t)), \quad d(\tilde{p}, \tilde{T}(p, q)(t)) = t.$$

Since

$$\begin{aligned} & (d(\tilde{o}, \tilde{R}(p, q)(t)), d(\tilde{q}, \tilde{R}(p, q)(t))) \\ &= \tilde{G}_{\tilde{q}}(\tilde{R}(p, q)(t)) \\ &= F_q(T(p, q)(d(p, q) - t)) \\ &= (d(o, T(p, q)(d(p, q) - t)), d(q, T(p, q)(d(p, q) - t))), \end{aligned}$$

we have

$$r(\tilde{R}(p, q)(t)) = d(o, T(p, q)(d(p, q) - t)), \quad d(\tilde{q}, \tilde{R}(p, q)(t)) = t.$$

Thus we have (3) and

$$d(\tilde{p}, \tilde{T}(p, q)(t)) + d(\tilde{q}, \tilde{R}(p, q)(d(p, q) - t)) = d(p, q)$$

which proves (4). Then (5) follows from

$$\begin{aligned} r(\tilde{T}(p, q)(t)) &= d(o, T(p, q)(t)) \\ &= r(\tilde{R}(p, q)(d(p, q) - t)). \end{aligned}$$

Obviously, (6) follows from the definition of the reference curves and the reference reverse curves \square

Let $q(t) = T(p, q)(t)$ and $\tilde{q}(t) = \tilde{T}(p, q)(t)$, $0 \leq t \leq d(p, q)$. Lemma 16 (3) shows that a geodesic triangle $\triangle \tilde{o}\tilde{p}\tilde{q}(t)$ in \tilde{M} is a comparison triangle corresponding to $\triangle opq(t)$ in M .

Let $\theta(t) = \theta(\tilde{R}(p, q)(d(p, q) - t)) - \theta(\tilde{T}(p, q)(t))$, $0 \leq t \leq d(p, q)$. The following lemma shows the difference between $\tilde{T}(p, q)(t)$ and $\tilde{R}(p, q)(t)$ in terms of $\theta(t)$.

Lemma 17. *The reference curves $\tilde{q}(t) = \tilde{T}(p, q)(t)$ and $\tilde{R}(p, q)(t)$ satisfy the following properties.*

- (1) $\theta(t) \geq 0$ for all $t \in [0, d(p, q)]$. Moreover, if $\theta(t) \neq 0$ at $t \in (0, d(p, q))$, namely $\tilde{T}(p, q)(t) \neq \tilde{R}(p, q)(d(p, q) - t)$, then $T(\tilde{p}, \tilde{q})$ does not cross the subarc of the parallel $[r = r(\tilde{q}(t))]$ in $\tilde{M}_{\tilde{p}}^+$ joining $\tilde{T}(p, q)(t)$ and $\tilde{R}(p, q)(d(p, q) - t)$.
- (2) $\theta(t) = 0$ if and only if $\tilde{T}(p, q)(t) = \tilde{R}(p, q)(d(p, q) - t)$. Then the point is in a minimizing geodesic segment $T(\tilde{p}, \tilde{q})$.
- (3) If there exists a point $\tilde{q}' \in \tilde{T}(p, q) \cap T(\tilde{p}, \tilde{q}) \setminus \{\tilde{p}, \tilde{q}\}$ (resp., $\tilde{R}(p, q) \cap T(\tilde{p}, \tilde{q}) \setminus \{\tilde{p}, \tilde{q}\}$), then $\tilde{R}(p, q)(d(p, q) - d(\tilde{p}, \tilde{q}')) = \tilde{q}'$ (resp., $\tilde{T}(p, q)(d(p, q) - d(\tilde{p}, \tilde{q}')) = \tilde{q}'$).
- (4) $\tilde{T}(p, q) \geq T(\tilde{p}, \tilde{q})$ if and only if $\tilde{R}(p, q) \geq T(\tilde{p}, \tilde{q})$.

(5) $\tilde{T}(p, q) \leq T(\tilde{p}, \tilde{q})$ if and only if $\tilde{R}(p, q) \leq T(\tilde{p}, \tilde{q})$.

Proof. It follows that $\theta(\tilde{p}) < \theta(\tilde{T}(p, q)(t))$ and $\theta(\tilde{p}) < \theta(\tilde{R}(p, q)(d(p, q) - t))$, $0 < t < d(p, q)$. Suppose the first part of (1) is false. Then Lemma 16 (5) and Lemma 11 (1) show that

$$d(\tilde{p}, \tilde{T}(p, q)(t)) > d(\tilde{p}, \tilde{R}(p, q)(d(p, q) - t))$$

for some t . This contradicts Lemma 16 (4), since

$$\begin{aligned} d(\tilde{p}, \tilde{q}) &= d(\tilde{p}, \tilde{T}(p, q)(t)) + d(\tilde{q}, \tilde{R}(p, q)(d(p, q) - t)) \\ &> d(\tilde{p}, \tilde{R}(p, q)(d(p, q) - t)) + d(\tilde{q}, \tilde{R}(p, q)(d(p, q) - t)) \\ &\geq d(\tilde{p}, \tilde{q}). \end{aligned}$$

We prove the second part of (1). Suppose that there exists a point $T(\tilde{p}, \tilde{q})(t_0)$ lying on the parallel circle joining $\tilde{q}(t)$ and $\tilde{R}(p, q)(d(p, q) - t)$ for some $t \in [0, d(p, q)]$. We then have $r(T(\tilde{p}, \tilde{q})(t_0)) = r(\tilde{q}(t))$. Since $\theta(\tilde{T}(p, q)(t))$ is monotone increasing in $t \in [0, d(p, q)]$, we have $\theta(\tilde{p}) \leq \theta(\tilde{q}(t))$ and $\theta(\tilde{R}(p, q)(d(p, q) - t)) \leq \theta(\tilde{q})$. Since $\theta(\tilde{q}(t)) < \theta(T(\tilde{p}, \tilde{q})(t_0)) < \theta(\tilde{R}(p, q)(d(p, q) - t))$, it follows from Lemma 11 (1) that $t < t_0$ and $d(\tilde{p}, \tilde{q}) - t_0 > d(p, q) - t$. Hence, we have $d(\tilde{p}, \tilde{q}) > d(p, q)$, a contradiction.

If $\theta(t) = 0$, then the equality holds in the above inequalities, and hence $\tilde{T}(p, q)(t) = \tilde{R}(p, q)(d(p, q) - t)$. The converse is trivial. The second part of (2) follows from Lemma 16 (4).

We prove (3). Let $q' \in T(p, q)$ correspond to \tilde{q}' , namely, $d(p, q') = d(\tilde{p}, \tilde{q}')$. Recall that $\tilde{p}' := \tilde{R}(p, q)(d(p, q) - d(p, q'))$ is the point in $\widetilde{M_q^-}$ such that $d(\tilde{o}, \tilde{p}') = d(o, q')$ and $d(\tilde{q}, \tilde{p}') = d(q, q')$. Since $d(q, q') = d(p, q) - d(p, q') = d(\tilde{p}, \tilde{q}) - d(\tilde{p}, \tilde{q}') = d(\tilde{q}, \tilde{q}')$ and $d(\tilde{o}, \tilde{q}') = d(o, q')$, we have $\tilde{p}' = \tilde{q}'$. In the same way we can prove the other case.

For the proof of (4) and (5), we suppose for indirect proof that $\tilde{T}(p, q) \geq T(\tilde{p}, \tilde{q})$ and $\tilde{R}(p, q)(s) < T(\tilde{p}, \tilde{q})$ for some $s \in (0, d(p, q))$. Then, there exists a point $\tilde{z} \in T(\tilde{p}, \tilde{q})$ such that $\theta(\tilde{T}(p, q)(d(p, q) - s)) \leq \theta(\tilde{z}) < \theta(\tilde{R}(p, q)(s))$, contradicting the second part of (1). The remainder cases are proved in the same way. \square

Lemma 18. *Let $q(t) = T(p, q)(t)$ be a minimizing geodesic segment in M . Assume that $\tilde{T}(p, q(t)) \geq T(\tilde{p}, \tilde{q}(t))$ for any $t \in [0, d(p, q)]$. Set $\tilde{p}(t) = \tilde{R}(p, q)(t)$. Then, we have*

- (1) $T(\tilde{p}, \tilde{q}(t)) \geq T(\tilde{p}, \tilde{q}(s))$ and $T(\tilde{p}(t), \tilde{q}) \geq T(\tilde{p}(s), \tilde{q})$ for any $t < s$.
- (2) $r(\tilde{q}(t)) \geq r(T(\tilde{p}, \tilde{q})(t))$ and $r(\tilde{p}(t)) \geq r(T(\tilde{p}, \tilde{q})(d(p, q) - t))$, $0 \leq t \leq d(p, q)$.
- (3) $\angle opq \geq \angle \tilde{o}\tilde{p}\tilde{q}$ and $\angle oqp \geq \angle \tilde{o}\tilde{q}\tilde{p}$.

Proof. We notice that $\tilde{T}(p, q(t)) \cup T(\tilde{p}, \tilde{q}(t))$ bounds a figure Ω in \widetilde{M} . We see that $T(\tilde{p}, \tilde{q}(s))$ cannot pass through any interior point of Ω . In fact, if $T(\tilde{p}, \tilde{q}(s))$ contains an interior point in Ω , then $T(\tilde{p}, \tilde{q}(s))$ meets $\tilde{T}(p, q(t))$ at $\tilde{T}(p, q(t))(t_0)$ for some $t_0 \in (0, t]$, because $T(\tilde{p}, \tilde{q}(t)) \cap T(\tilde{p}, \tilde{q}(s)) = \{\tilde{p}\}$ and $\tilde{q}(s) \notin \Omega$. Since $\tilde{T}(p, q(t))$ is a subarc of $\tilde{T}(p, q(s))$, we have $\tilde{T}(p, q(t)) \geq \tilde{T}(p, q(s))$ and, hence, $\tilde{T}(p, q(t)) \geq T(\tilde{p}, \tilde{q}(s))$. This means that the last parameter t_0 where $T(\tilde{p}, \tilde{q}(s))$ meets $\tilde{T}(p, q(t))$ must be t . This contradicts $T(\tilde{p}, \tilde{q}(t)) \cap T(\tilde{p}, \tilde{q}(s)) = \{\tilde{p}\}$ again. Since $\tilde{T}(p, q(t)) \geq T(\tilde{p}, \tilde{q}(s))$, we conclude that $T(\tilde{p}, \tilde{q}(t)) \geq T(\tilde{p}, \tilde{q}(s))$. By Lemma 17 (4), we have the same inequality for the reference reverse curves. This completes the proof of (1).

Given t , $0 \leq t \leq d(p, q)$, we set $c(s) = T(\tilde{p}, \tilde{q}(s))(t) = S(\tilde{p}, t) \cap T(\tilde{p}, \tilde{q}(s))$ for any $s \in (t, d(p, q)]$. Then, (1) implies that $r(c(s))$ is monotone nonincreasing for $s > t$. We then have

$$r(\tilde{q}(t)) \geq r(c(s)) \geq r(c(d(p, q))) = r(T(\tilde{p}, \tilde{q})(t)).$$

In the same way, we have $r(\tilde{p}(t)) \geq r(T(\tilde{p}, \tilde{q})(d(p, q) - t))$ for any t . This completes the proof of (2).

In order to prove (3) we recall that

$$\cos \angle opq = \lim_{t \rightarrow +0} \frac{d(p, q(t))^2 + d(o, p)^2 - d(o, q(t))^2}{2d(p, q(t))d(o, p)}.$$

Therefore, we have from (2)

$$\begin{aligned} \cos \angle opq &= \lim_{t \rightarrow +0} \frac{t^2 + r(\tilde{p})^2 - r(\tilde{q}(t))^2}{2tr(\tilde{p})} \\ &\leq \lim_{t \rightarrow +0} \frac{t^2 + r(\tilde{p})^2 - r(T(\tilde{p}, \tilde{q})(t))^2}{2tr(\tilde{p})} = \cos \angle \tilde{o}\tilde{p}\tilde{q}. \end{aligned}$$

Using the reference reverse curve $\tilde{R}(p, q)$, we have $\angle oqp \geq \angle \tilde{o}\tilde{q}\tilde{p}$ in the same way. This completes the proof of (3). \square

5. REFERENCE CURVES MEETING NO CUT POINT

In this section, we assume that a complete pointed Riemannian manifold (M, o) is referred to a surface of revolution $(\widetilde{M}, \tilde{o})$. When $\ell < \infty$, it has been proved in [2] that M is isometric to the warped product manifold whose warping function is the radial curvature function of \widetilde{M} if there exists a point $p \in M$ such that $d(o, p) = \ell$. Hence, there is nothing to study for the comparison theorems on those manifolds anymore. Therefore, we may assume that $d(o, p) < \ell$ for all points $p \in M$. We study the global positional relation between reference curves $\tilde{T}(p, q)$

and minimizing geodesic segments $T(\tilde{p}, \tilde{q})$. We start from the following lemma, showing the local relation, which is proved in [2], [3] and [5].

Lemma 19. *Let p be a point in M such that $p \neq o$. There exists an $r_p > d(o, p)$ such that any geodesic triangle $\triangle opq$ in M with $d(o, q) + d(p, q) < r_p$ has a comparison triangle $\triangle \tilde{o}\tilde{p}\tilde{q}$ in \widetilde{M} satisfying (2.2) and (2.3). Moreover, if one of the equalities holds in (2.2) and (2.3), then $\triangle opq$ bounds a totally geodesic 2-dimensional submanifold in M which is isometric to a comparison triangle domain $\triangle \tilde{o}\tilde{p}\tilde{q}$ corresponding to $\triangle opq$.*

Proof. As was seen in Lemma 12 (2), the set of all ellipses $E(\tilde{o}, \tilde{p}; a)$, $a > d(\tilde{o}, \tilde{p})$, gives a foliation of $\widetilde{M} \setminus T(\tilde{o}, \tilde{p})$. Namely, for any point $\tilde{q} \in \widetilde{M} \setminus T(\tilde{o}, \tilde{p})$ there exists the unique ellipse $E(\tilde{o}, \tilde{p}; a)$ passing through \tilde{q} . When $r(\tilde{p}) < \ell$, there exists a positive δ such that the δ -neighborhood $D(\delta)$ of $T(\tilde{o}, \tilde{p})$ does not contain any cut point of \tilde{p} . Observe that the proof of the comparison theorems in [2], [3] and [5] is valid if the domain is free from $Cut(\tilde{p})$. Hence, if we set $r_p = \max\{a \mid E(\tilde{o}, \tilde{p}; a) \subset D(\delta)\}$, then it satisfies this lemma. \square

It follows from (2.2) and the third inequality of (2.3) that the reference curves and the comparison triangle $\triangle \tilde{o}\tilde{p}\tilde{q}$ actually lie in \widetilde{M}_p^+ for all points q with $d(o, q) + d(p, q) < r_p$.

Corollary 20. *Let p and q be points in M other than o . Assume that a minimizing geodesic segment $T(p, q)$ is contained in $F_p^{-1}(\widetilde{F}_p(\widetilde{M}_p^+))$. If $\widetilde{T}(p, q) = T(\tilde{p}, \tilde{q})$ as a set, then $\triangle opq$ bounds a totally geodesic 2-dimensional submanifold in M which is isometric to a comparison triangle domain $\triangle \tilde{o}\tilde{p}\tilde{q}$ corresponding to $\triangle opq$.*

Proof. As before, set $q(t) = T(p, q)(t)$ and $\tilde{q}(t) = \widetilde{T}(p, q)(t)$, $0 \leq t \leq d(p, q)$. Since $d(\tilde{p}, \tilde{q}(t)) = t$, $0 \leq t \leq d(p, q)$, and $\widetilde{T}(p, q) = T(\tilde{p}, \tilde{q})$, we have $\tilde{q}(t) = T(\tilde{p}, \tilde{q})(t)$ for all t . In fact, if $\tilde{q}(t) \neq T(\tilde{p}, \tilde{q})(t)$ for some $t \in (0, d(p, q))$, then there exists t_0 such that $t_0 \neq t$ and $\tilde{q}(t) = T(\tilde{p}, \tilde{q})(t_0)$. We then have $t = d(p, q(t)) = d(\tilde{p}, \tilde{q}(t)) = d(\tilde{p}, T(\tilde{p}, \tilde{q})(t_0)) = t_0$, a contradiction. Hence, if $0 \leq t < s \leq d(p, q)$, we then have $\widetilde{T}(q(t), q(s))(s - t) = T(\tilde{q}(t), \tilde{q}(s))(s - t)$, since $d(\tilde{q}(t), \tilde{q}(s)) = s - t = d(q(t), q(s))$, $r(q(t)) = r(\tilde{q}(t))$ and $r(q(s)) = r(\tilde{q}(s))$. Hence, we have $\angle oq(t)q(s) = \angle \tilde{o}\tilde{q}(t)\tilde{q}(s)$. It follows from Lemma 19 that there exists a $\delta > 0$ such that if $|s - t| < \delta$, then $\triangle oq(t)q(s)$ bounds a totally geodesic 2-dimensional submanifold in M which is isometric to a comparison triangle domain $\triangle \tilde{o}\tilde{q}(t)\tilde{q}(s)$ corresponding to $\triangle oq(t)q(s)$. This shows that there exist a totally geodesic 2-dimensional submanifold Δ bounded by

$\triangle opq$ and an isometry from \triangle onto the domain bounded by $\triangle \tilde{o}\tilde{p}\tilde{q}$ in \widetilde{M} . \square

Remark 21. Our proof technique to be employed in the theorems makes it complicated to treat the case where $\tilde{T}(p, q) = T(\tilde{p}, \tilde{q})$. In order to avoid the case, we employ the same ideas developed in Chapter 2 in [1].

Let $K(r)$, $r \in [0, \ell]$, denote the Gauss curvature of \widetilde{M} on the parallel r -circle. For a sufficiently small $\delta > 0$ we consider a differential equation

$$f''(r) + (K(r) - \delta)f(r) = 0.$$

We denote by $f_\delta(r)$ its solution with $f_\delta(0) = 0$ and $f'_\delta(0) = 1$. Then, $f_\delta(r) > f(r)$ for any $r \in (0, \ell)$. By defining a metric to be

$$ds^2 = dr^2 + f_\delta(r)^2 d\theta^2,$$

we have a surface of revolution \widetilde{M}_δ such that M is referred to \widetilde{M}_δ . When $\ell < \infty$, the coefficient $K(r) - \delta$ and the solution $f_\delta(r)$ are extended on an interval $[0, \ell']$ containing $[0, \ell]$ properly and we do not assume that $f_\delta(\ell') = 0$ and $f'_\delta(\ell') = -1$. To avoid the confusing case where some equality holds in (2.2) or (2.3), we employ \widetilde{M}_δ instead of \widetilde{M} . We prove our results by thinking of \widetilde{M}_δ as the reference surface, and then conclude the proof by letting $\delta \rightarrow 0$. More precisely, we choose $\delta = \delta(R)$ for each R with $0 < (R + d(o, p))/2 < \ell$ such that (2.1) holds in the insides of $E(o, p; R)$ and $E(\tilde{o}, \tilde{p}; R)$ and such that $\delta(R)$ converges to 0 as $(R + d(o, p))/2 \rightarrow \ell$. We prove (2.2) and (2.3) in the interior of $B(o, p; R)$ and $B(\tilde{o}, \tilde{p}; R)$, and then take $(R + d(o, p))/2$ to ℓ . The most important fact is that $\tilde{T}(p, q) = T(\tilde{p}, \tilde{q})$ does not occur in \widetilde{M}_δ for any points $q \neq p$ in $E(o, p; R)$. This property simplifies our discussion.

The following lemma is proved in [3]. The proof here is different from theirs. Moreover, the method in the proof will be used when we prove Theorems in §7.

Lemma 22. *Assume that a point $q \in M$ admits a minimizing geodesic segment $T(p, q)$ contained in $F_p^{-1}(\widetilde{F}_{\tilde{p}}^+(\widetilde{M}_{\tilde{p}}^+))$. If there exists a point $q_1 \in T(p, q)$ such that $\tilde{T}(p, q') \geq T(\tilde{p}, \tilde{q}')$ for all $q' \in T(p, q_1)$, $\tilde{q}_1 \notin \text{Cut}(\tilde{p})$ and $((\tilde{T}(p, q) \setminus \{\tilde{q}\}) \setminus \tilde{T}(p, q_1)) \cap \text{Cut}(\tilde{p}) = \emptyset$, then there exists a minimizing geodesic segment $T(\tilde{p}, \tilde{q})$ such that $\tilde{T}(p, q) \geq T(\tilde{p}, \tilde{q})$. In addition, if $\tilde{T}(p, q) \cap T(\tilde{p}, \tilde{q})$ contains a point \tilde{q}' other than \tilde{p} and \tilde{q} , then $\triangle opq$ bounds a totally geodesic 2-dimensional submanifold in M which is isometric to a comparison triangle domain $\triangle \tilde{o}\tilde{p}\tilde{q}$ corresponding to $\triangle opq$ in \widetilde{M} .*

The point is if $\tilde{q}_1 \notin \text{Cut}(\tilde{p})$ or not. In case of $\tilde{q}_1 \notin \text{Cut}(\tilde{p})$ the reference curve can be extended, keeping the positional relation to a minimizing geodesic connecting its end points.

Proof. In order to prove the first part, we work in \widetilde{M}_δ to avoid the case where a reference curve is identified with a minimizing geodesic segment connecting its endpoints. For convenience we set $q(t) = T(p, q)(t)$ and $\tilde{q}(t) = \widetilde{T}(p, q)(t)$ for any $t \in (0, d(p, q))$. Let t_0 be the least upper bound of the set of all $t_2 \leq d(p, q)$ so that there exists a minimizing geodesic segment $T(\tilde{p}, \tilde{q}(t))$ with $\widetilde{T}(p, q(t)) \geq T(\tilde{p}, \tilde{q}(t))$ for all $t \in (0, t_2)$. If t_1 is the parameter such that $q_1 = q(t_1)$, we then have $t_0 \geq t_1$ because of the assumption.

Suppose for indirect proof that $t_0 < d(p, q)$. Since we assume that $\tilde{q}(t_0) \notin \text{Cut}(\tilde{p})$, there exists a neighborhood V of $\tilde{q}(t_0)$ such that $T(\tilde{p}, \tilde{x})$ is the unique minimizing geodesic segment connecting \tilde{p} and $\tilde{x} \in V$. Since the minimizing geodesic segment $T(\tilde{p}, \tilde{q}(t_0))$ is unique, $\widetilde{T}(p, q(t)) \geq T(\tilde{p}, \tilde{q}(t))$ for all $t \in (0, t_0)$ implies that $\widetilde{T}(p, q(t_0)) \geq T(\tilde{p}, \tilde{q}(t_0))$.

We will prove that there exists an $\varepsilon > 0$ such that $\widetilde{T}(p, q(t_0 + t)) \geq T(\tilde{p}, \tilde{q}(t_0 + t))$, $0 \leq t \leq \varepsilon$. Suppose that there exists a monotone decreasing sequence t_j converging to 0 such that no minimizing geodesic segment $T(\tilde{p}, \tilde{q}(t_0 + t_j))$ satisfies $\widetilde{T}(p, q(t_0 + t_j)) \geq T(\tilde{p}, \tilde{q}(t_0 + t_j))$. We then have either

$$\widetilde{T}(p, q(t_0 + t_j)) \leq T(\tilde{p}, \tilde{q}(t_0 + t_j))$$

or

$$\widetilde{T}(p, q(t_0 + t_j)) \cap T(\tilde{p}, \tilde{q}(t_0 + t_j)) \neq \{\tilde{p}, \tilde{q}(t_0 + t_j)\}.$$

Suppose the first is true. We then have $\widetilde{T}(p, q(t_0)) = T(\tilde{p}, \tilde{q}(t_0))$. In fact, since $T(\tilde{p}, \tilde{q}(t_0 + t_j))$ converges to $T(\tilde{p}, \tilde{q}(t_0))$ which is the unique minimizing geodesic segment connecting \tilde{p} and $\tilde{q}(t_0)$, we have $\widetilde{T}(p, q(t_0)) \leq T(\tilde{p}, \tilde{q}(t_0))$. Combining $\widetilde{T}(p, q(t_0)) \geq T(\tilde{p}, \tilde{q}(t_0))$, we conclude $\widetilde{T}(p, q(t_0)) = T(\tilde{p}, \tilde{q}(t_0))$. Since we employ \widetilde{M}_δ , this yields a contradiction because of Corollary 20.

Suppose the second is true. Let \tilde{q}_j be a point in $\widetilde{T}(p, q(t_0 + t_j)) \cap T(\tilde{p}, \tilde{q}(t_0 + t_j))$ such that it is different from \tilde{p} , $\tilde{q}(t_0 + t_j)$ and $T(\tilde{q}_j, \tilde{q}(t_0 + t_j)) \not\leq \widetilde{T}(p, q(t_0 + t_j))$. Let $q_j \in T(p, q(t_0 + t_j))$ be the point with $F_p(q_j) = \widetilde{F}_{\tilde{p}}(\tilde{q}_j)$. If \tilde{q}_j does not converge to the point $\tilde{q}(t_0)$, then there exists an accumulation point $\tilde{q}' \neq \tilde{q}(t_0)$ such that $\tilde{q}' \in T(\tilde{p}, \tilde{q}(t_0))$. This situation implies that $\widetilde{T}(p, q(t_0)) \geq T(\tilde{p}, \tilde{q}(t_0))$ and $\widetilde{T}(p, q(t_0)) \cap T(\tilde{p}, \tilde{q}(t_0)) \supset \{\tilde{p}, \tilde{q}', \tilde{q}(t_0)\}$, which is the assumption of the second part

of this lemma, to be proved in the next paragraph. This is impossible because we now work in \widetilde{M}_δ . We have proved that \tilde{q}_j converges to $\tilde{q}(t_0)$. We then have $\tilde{q}(t_0 + t_j)$ such that $\tilde{T}(p, q_j) \geq T(\tilde{p}, \tilde{q}_j)$ and $\tilde{T}(p, q(t_0 + t_j)) \setminus \tilde{T}(p, q_j) \not\geq T(\tilde{q}_j, \tilde{q}(t_0 + t_j))$, since there exists the unique minimizing geodesic segment $T(p, q_j)$ which is a subsegment of $T(p, q(t_0 + t_j))$. On the other hand, for sufficiently large j , it follows from Lemma 17 (3) that $\tilde{R}(p, q(t_0 + t_j))$ passes through \tilde{q}_j . From Lemma 16 (6) the reference reverse curve $\tilde{R}(q_j, q(t_0 + t_j))$ is a subarc of $\tilde{R}(p, q(t_0 + t_j))$ from $\tilde{q}(t_0 + t_j)$ to \tilde{q}_j which lies in the same side as the subarc of $\tilde{T}(p, q(t_0 + t_j))$ from \tilde{q}_j to $\tilde{q}(t_0 + t_j)$ (see Lemma 17 (4) and (5)). Thus, we have the positional relation

$$\tilde{R}(q_j, q(t_0 + t_j)) \leq T(\tilde{q}_j, \tilde{q}(t_0 + t_j)).$$

However, this contradicts Lemma 19 near the point $\tilde{q}(t_0 + t_j)$. We conclude that $t_0 = d(p, q)$ by employing \widetilde{M}_δ . Letting $\delta \rightarrow 0$ we complete the proof of the first part.

We prove the second part. If $\tilde{T}(p, q) \setminus \tilde{T}(p, q') \not\subset T(\tilde{p}, \tilde{q})$, then there exists a point $q'' \in T(q', q)$ such that \tilde{q}'' does not lie in $T(\tilde{p}, \tilde{q})$ and hence $\tilde{q}'' > T(\tilde{p}, \tilde{q})$. Therefore, $\tilde{q}' < T(\tilde{p}, \tilde{q}'')$, contradicting that $\tilde{T}(p, q'') \geq T(\tilde{p}, \tilde{q}'')$. Thus, we have $\tilde{T}(p, q) \setminus \tilde{T}(p, q') \subset T(\tilde{p}, \tilde{q})$, in other words, $\tilde{R}(q', q) \subset T(\tilde{p}, \tilde{q})$. Let u and u' be points in $T(p, q)$ such that they are near q' and p , u, q' and u' lie in this order in $T(p, q)$. If $\tilde{u} \notin T(\tilde{p}, \tilde{q})$, then we have a contradiction from Lemma 19 and the same argument above. Thus, the segment $T(q', q)$ satisfying $\tilde{R}(q', q) \subset T(\tilde{p}, \tilde{q})$ can be extended until q' reaches p . Hence, we have $\tilde{R}(p, q) = T(\tilde{p}, \tilde{q})$, and, equivalently, $\tilde{T}(p, q) = T(\tilde{p}, \tilde{q})$. It follows from Corollary 20 that $\triangle opq$ bounds a totally geodesic 2-dimensional submanifold in M which is isometric to a comparison triangle domain $\triangle \tilde{op}\tilde{q}$ corresponding to $\triangle opq$ in \widetilde{M} . \square

6. REFERENCE CURVES MEETING CUT POINTS

For two points $\tilde{x}, \tilde{y} \in \widetilde{M}_p^+$ with $\theta(\tilde{x}) \neq \theta(\tilde{y})$, let $U(\tilde{x}, \tilde{y})$ and $L(\tilde{x}, \tilde{y})$ denote the minimizing geodesic segments joining \tilde{x} to \tilde{y} such that

$$U(\tilde{x}, \tilde{y}) \geq T(\tilde{x}, \tilde{y}) \geq L(\tilde{x}, \tilde{y}), \quad \text{for all } T(\tilde{x}, \tilde{y})$$

Notice that $U(\tilde{x}, \tilde{y}) = L(\tilde{x}, \tilde{y})$ if and only if $\tilde{y} \notin \text{Cut}(\tilde{x})$ or $\tilde{y} \in \text{Cut}(\tilde{x})$ is an end point of $\text{Cut}(\tilde{x})$ such that \tilde{y} is an isolated conjugate point to \tilde{x} along the unique minimizing geodesic. The following lemma is a consequence of Lemma 13 and plays an important role for the proof of our Theorems.

Lemma 23. *Assume that $B(o, p; r) \subset F_p^{-1}(\widetilde{F}_{\tilde{p}}(\widetilde{M}_{\tilde{p}}^+))$. If $q \in E(o, p; r)$ is not a local maximum point of d_r on $E(o, p; r)$, then there exists a sequence of points $q_j \in E(o, p; r)$ converging to q such that $T(\tilde{p}, \tilde{q}_j)$ converges to $U(\tilde{p}, \tilde{q})$ as $j \rightarrow \infty$. In particular, if $U(\tilde{p}, \tilde{q}) \neq L(\tilde{p}, \tilde{q})$, then any extension of $U(\tilde{p}, \tilde{q})$ crosses $Cut(\tilde{p})$ from the far side of \tilde{o} to the near side of \tilde{o} .*

Proof. Let q_j be a sequence of points in $E(o, p; r)$ converging to q such that $d(o, q_j) > d(o, q)$ for all j . Then we have

$$\begin{aligned} d(p, q_j) &= r - d(o, q_j) \\ &< r - d(o, q) = d(p, q). \end{aligned}$$

In view of Lemma 12, we observe that $T(\tilde{p}, \tilde{q}_j)$ does not cross $[\theta(\tilde{q}) \leq \theta] \cap [r = d(o, q)]$. This means that $T(\tilde{p}, \tilde{q}_j) \setminus \{\tilde{p}\} > T(\tilde{p}, \tilde{q}) \setminus \{\tilde{p}\}$ for every minimizing geodesic segment $T(p, q)$. Therefore, $T(\tilde{p}, \tilde{q}_j)$ converges to $U(\tilde{p}, \tilde{q})$ as $j \rightarrow \infty$. \square

We observe from Lemma 22 that $\tilde{T}(p, q) \geq U(\tilde{p}, \tilde{q})$ if $\tilde{T}(p, q) \setminus \{\tilde{q}\} \cap Cut(\tilde{p}) = \emptyset$ and $q \in E(o, p; r)$ is not a local maximum point of d_r .

In the proof of the following lemma, we need an orientation of the intersection points of curves and $Cut(\tilde{p})$. Let $\tilde{x} \in Cut(\tilde{p})$. A curve $c(\theta)$, $\tilde{x} = c(\theta_0)$, parameterized by angle coordinate θ is said to intersect $Cut(\tilde{p})$ *positively* (resp., *negatively*) at a point $\tilde{x} = c(\theta_0)$ if there is a small neighborhood Ω around \tilde{x} such that $c \cap \Omega \geq Cut(\tilde{p}) \cap \Omega$ for $\theta \leq \theta_0$, (resp., $c \cap \Omega \leq Cut(\tilde{p}) \cap \Omega$ for $\theta \leq \theta_0$). Intuitively, "intersecting positively" means that c meets $Cut(\tilde{p})$ from the far side with respect to \tilde{o} .

Lemma 24. *Let $q \in M$ and let $T(p, q)$ be a minimizing geodesic segment. Assume that $T(p, q) \subset F_p^{-1}(\widetilde{F}_{\tilde{p}}(\widetilde{M}_{\tilde{p}}^+))$. Suppose all intersection points of $\tilde{T}(p, q)$ and $Cut(\tilde{p})$ are positive. Then, we have*

$$\tilde{T}(p, q(t)) \geq U(\tilde{p}, \tilde{q}(t)), \quad 0 \leq t < d(p, q).$$

Here we set $q(t) = T(p, q)(t)$, $0 \leq t \leq d(p, q)$.

Notice again that if $q \notin Cut(p)$, then there exists a unique minimizing geodesic segment $T(p, q)$, and hence, the reference curve $\tilde{T}(p, q)$ is uniquely determined. However, this does not mean that the reference curve connecting \tilde{p} and \tilde{q} uniquely exists, because $F_p^{-1}(\widetilde{F}_{\tilde{p}}(\tilde{q}))$ may not be a single point. If $\tilde{q} \notin Cut(\tilde{p})$, then there exists a unique minimizing geodesic segment $T(\tilde{p}, \tilde{q})$, and hence, $T(\tilde{p}, \tilde{q}) = U(\tilde{p}, \tilde{q}) = L(\tilde{p}, \tilde{q})$. However, if $\tilde{q} \in Cut(\tilde{p})$, then there may be many minimizing geodesic

segments $T(\tilde{p}, \tilde{q})$. So the positional relation between $\tilde{T}(p, q)$ and $U(\tilde{p}, \tilde{q})$ is unknown, in general. These facts are often used without notice.

Proof. We work in \widetilde{M}_δ instead of the reference surface \widetilde{M} . We choose δ to be sufficiently small so that \widetilde{M}_δ satisfies the assumption in this lemma. Let t_0 be the least upper bound of the set of all $t_1 \in (0, d(p, q))$ such that $\tilde{T}(p, q(t)) \geq U(\tilde{p}, \tilde{q}(t))$ for all $t \in (0, t_1)$. We already know that $t_0 > 0$. Suppose for indirect proof that $t_0 < d(p, q)$. If $\tilde{q}(t_0) \notin \text{Cut}(\tilde{p})$, then, from Lemma 22, there exists an $\varepsilon > 0$ such that $\tilde{T}(p, q(t_0 + t)) \geq U(\tilde{p}, \tilde{q}(t_0 + t))$ for all $t \in (0, \varepsilon)$. This contradicts the choice of t_0 .

Suppose $\tilde{q}(t_0) \in \text{Cut}(\tilde{p})$. Since the minimizing geodesic segment $T(p, q(t_0))$ is unique and $\tilde{q}(t_0)$ is a positive cut point, it follows that $\tilde{T}(p, q(t_0)) \geq U(\tilde{p}, \tilde{q}(t_0))$. We prove that there exists an $\varepsilon > 0$ such that $\tilde{T}(p, q(t_0 + t)) \leq \text{Cut}(\tilde{p})$ for all $t \in (0, \varepsilon)$. In fact, suppose this is not true. Then, there exist a sufficiently small neighborhood Ω around $\tilde{q}(t_0)$ and a sequence $t_j > t_0$ such that t_j converges to t_0 and $\tilde{q}(t_j)$ is contained in the subdomain of Ω bounded below by $U(\tilde{p}, \tilde{q}(t_0)) \cup \text{Cut}(\tilde{p})$. Let T_j be a minimizing geodesic segment connecting \tilde{p} and $\tilde{q}(t_j)$. Then, we know that $T_j \cap \tilde{T}(p, q(t_j)) \neq \{\tilde{p}, \tilde{q}(t_j)\}$, since T_j converges to $U(\tilde{p}, \tilde{q}(t_0))$ and $\tilde{T}(p, q)([0, t_0]) \neq T(\tilde{p}, \tilde{q}(t_0))$. If $\tilde{q}(t'_j) = \tilde{T}(p, q)(t'_j) \in T_j \cap \tilde{T}(p, q(t_j))$ and $\tilde{q}(t'_j) \notin \{\tilde{p}, \tilde{q}(t_j)\}$, then $\tilde{q}(t'_j)$ converges to $\tilde{q}(t_0)$ and it follows that

$$\tilde{T}(p, q)([t'_j, t_j]) \leq T(\tilde{q}(t'_j), \tilde{q}(t_j)).$$

However, this contradicts Lemma 19 and Lemma 17. Thus we see that there exists an $\varepsilon > 0$ such that $\tilde{T}(p, q(t_0 + t)) \leq \text{Cut}(\tilde{p})$ for all $t \in (0, \varepsilon)$, and, therefore, $\tilde{q}(t_0 + t) \notin \text{Cut}(\tilde{p})$ for all small $t > 0$. Since $a := d(o, q(t_0)) + d(p, q(t_0)) < d(o, q(t_0 + t)) + d(p, q(t_0 + t))$, the point $\tilde{q}(t_0 + t)$ is outside $E(o, p; a)$. Therefore, $\tilde{q}(t_0 + t)$ is in the subdomain of Ω bounded above by $L(\tilde{p}, \tilde{q}(t_0)) \cup \text{Cut}(\tilde{p})$. Since a sequence of unique minimizing geodesic segments $T(\tilde{p}, \tilde{q}(t_0 + t))$ converges to $L(\tilde{p}, \tilde{q})$ as $t \rightarrow 0$, it follows that $\tilde{T}(p, q(t_0 + t)) \geq U(\tilde{p}, \tilde{q}(t_0 + t))$ for all small $t > 0$. This contradicts the choice of t_0 . \square

7. PROOF OF THEOREMS

We are ready to prove Theorems 5 and 7.

Proof of Theorems 5 and 7. Let r_0 be the least upper bound of the set of all $r_1 > d(o, p)$ satisfying the following properties: Let $r \in (d(o, p), r_1)$ and $q \in E(o, p; r)$. Then,

(C1) there exists a minimizing geodesic segment $T(p, q)$ such that $T(p, q)$ is contained in the set $F_p^{-1}(\widetilde{F}_{\tilde{p}}(\widetilde{M}_{\tilde{p}}^+))$ and $\widetilde{T}(p, q) \geq U(\tilde{p}, \tilde{q})$,

and

(C2) every minimizing geodesic segment $T(p, q)$ is contained in the set $F_p^{-1}(\widetilde{F}_{\tilde{p}}(\widetilde{M}_{\tilde{p}}^+))$ and satisfies $\widetilde{T}(p, q) \geq L(\tilde{p}, \tilde{q})$.

As was seen in Lemma 19, we have $r_0 > d(o, p)$. Let r be such that $d(o, p) < r < r_0$. Let $q \in E(o, p; r)$, $q_1(t) = T(o, q)(t)$ and $\tilde{q}_1(t) = \widetilde{T}(o, q)(t)$ for any $t \in [0, d(o, q)]$. Then, we have

$$\begin{aligned} & d(o, q_1(t)) + d(p, q_1(t)) \\ &= d(o, q) - d(q, q_1(t)) + d(p, q_1(t)) \\ &\leq d(o, q) + d(p, q) = r < r_0 \end{aligned}$$

for any $t \in (0, d(o, q))$, and hence, from the condition (C2), every $\triangle opq_1(t)$ in M has a comparison triangle $\triangle \tilde{o}\tilde{p}\tilde{q}_1(t)$ in $\widetilde{M}_{\tilde{p}}^+$ satisfying (2.2). Moreover, from the condition (C1), there exists a minimizing geodesic segment $T(p, q_1(t))$ such that $\widetilde{T}(p, q_1(t)) \geq U(\tilde{p}, \tilde{q}_1(t))$.

Assertion 25. *Let $q \in E(o, p; r)$ with $d(o, p) < r < r_0$. Assume that the minimizing geodesic segments $T(p, q)$ and $T(\tilde{p}, \tilde{q})$ satisfy $\widetilde{T}(p, q) \geq T(\tilde{p}, \tilde{q})$. Then for any minimizing geodesic segments $T(o, p)$ and $T(o, q)$ the geodesic triangle $\triangle opq = T(o, p) \cup T(p, q) \cup T(o, q)$ has the comparison triangle $\triangle \tilde{o}\tilde{p}\tilde{q}$ with edge $T(\tilde{p}, \tilde{q})$ which satisfies (2.3).*

Proof. It follows from Lemmas 18 (3), 19 and Corollary 20 that $\angle opq \geq \angle \tilde{o}\tilde{p}\tilde{q}$ and $\angle oqp \geq \angle \tilde{o}\tilde{q}\tilde{p}$ where one of two equalities holds if and only if the geodesic triangle $\triangle opq$ bounds a totally geodesic 2-dimensional submanifold in M which is isometric to a comparison triangle domain $\triangle \tilde{o}\tilde{p}\tilde{q}$ corresponding to $\triangle opq$ in \widetilde{M} , because $\widetilde{T}(p, q) = T(\tilde{p}, \tilde{q})$.

In order to show $\angle poq \geq \angle \tilde{p}\tilde{o}\tilde{q}$, we employ $\widetilde{M}_{\delta(r_0)}$ instead of \widetilde{M} and prove that $\theta(t) := \theta(\tilde{q}_1(t)) = \angle \tilde{p}\tilde{o}\tilde{q}_1(t)$ is monotone non-increasing in $t \in [0, d(\tilde{o}, \tilde{q})]$. Let $g_t(s) = d(p, q_1(t+s))$ and $\tilde{g}_t(s) = d(\tilde{p}, T(\tilde{o}, \tilde{q}_1(t))(t+s))$ for sufficiently small $s > 0$. Here, since $T(\tilde{o}, \tilde{q}_1(t))$ lies in the meridian through $\tilde{q}_1(t)$, we can define the point $T(\tilde{o}, \tilde{q}_1(t))(t+s)$ for any $s \in [-t, \ell - t]$. It follows from the conditions (C1), (C2) and Lemma 18 that $\pi \geq \angle oq_1(t)p > \angle \tilde{o}\tilde{q}_1(t)\tilde{p}$ for every $t \in (0, d(o, q))$. Let $g'_{t+}(0)$ denote the right hand derivative at $s = 0$, namely,

$$g'_{t+}(0) = \lim_{h \rightarrow 0+0} \frac{g_t(h) - g_t(0)}{h}.$$

If $\alpha(t)$ is the angle of $T(\tilde{o}, \tilde{q}_1(t))$ with $U(\tilde{p}, \tilde{q}_1(t))$, then the first variation formula implies that $\tilde{g}'_{t+}(0) = \cos \alpha(t)$ for every $t \in (0, d(o, q))$. Hence, we have $g'_{t+}(0) < \tilde{g}'_{t+}(0)$ because of the condition (C1) and Lemma 17 (4). There exists an $\varepsilon > 0$ such that

$$\begin{aligned} d(\tilde{p}, \tilde{q}_1(t+s)) &= d(p, q_1(t+s)) = g_t(s) \\ &< \tilde{g}_t(s) = d(\tilde{p}, T(\tilde{o}, \tilde{q}_1(t))(t+s)) \end{aligned}$$

for all $s \in (0, \varepsilon)$. Since $r(\tilde{q}_1(t+s)) = r(T(\tilde{o}, \tilde{q}_1(t))(t+s)) = t+s$, $\theta(T(\tilde{o}, \tilde{q}_1(t))(t+s)) = \theta(T(\tilde{o}, \tilde{q}_1(t))(t)) =: \theta(t)$, and $\theta(t+s) := \theta(\tilde{q}_1(t+s))$, it follows from Lemma 11 (2) that $\theta(t) > \theta(t+s)$ for all $s \in (0, \varepsilon)$. Thus, we have

$$\angle poq = \angle \tilde{p}\tilde{o}\tilde{q}_1(0) > \angle \tilde{p}\tilde{o}\tilde{q}_1(d(o, q)) = \angle \tilde{p}\tilde{o}\tilde{q}.$$

Thus, we have $\angle poq \geq \angle \tilde{p}\tilde{o}\tilde{q}$, employing the reference surface \widetilde{M} as $\delta(r_0)$ goes to 0. Here, the equality holds if and only if there exists a geodesic triangle $\triangle opq$ such that it bounds a totally geodesic 2-dimensional submanifold in M which is isometric to a comparison triangle domain $\triangle \tilde{o}\tilde{p}\tilde{q}$ corresponding to $\triangle opq$ in \widetilde{M} with edge $L(\tilde{p}, \tilde{q})$. \square

Assertion 26. *If $E(o, p; r_0) \neq \emptyset$, then every point $q \in E(o, p; r_0)$ satisfies that the conditions (C1) and (C2). In particular, $B(o, p; r_0) \subset F_p^{-1}(\widetilde{F}_{\tilde{p}}(\widetilde{M}_{\tilde{p}}^+))$.*

Proof. We first prove that for every point $q \in E(o, p; r_0)$ any minimizing geodesic segment $T(p, q)$ satisfies $\widetilde{T}(p, q) \geq L(\tilde{p}, \tilde{q})$. Suppose $\angle oqp \neq \pi$. Let $q_j \in T(p, q) \setminus \{p, q\}$ be a sequence of points converging to q . Then, $r(q_j) < r_0$ is satisfied. Hence, it follows from the definition of r_0 that $\widetilde{T}(p, q) \geq L(\tilde{p}, \tilde{q})$ holds as the limit of $\widetilde{T}(p, q_j) \geq L(\tilde{p}, \tilde{q}_j)$.

When $\angle oqp = \pi$, it is possible that there is no sequence of points q_j with $r(q_j) < r_0$ such that $q_j \rightarrow q$ (see Example 15). In this case, there exists a cut point p' (resp., o') of o (resp., p) in $T(p, q) \setminus \{p, q\}$ (resp., $T(o, q) \setminus \{o, q\}$). In particular, there exists the unique minimizing geodesic segment $T(p, q)$ connecting p and q and $T(p, q) \supset T(p, p')$. Since any point $q_j \in T(p, p') \setminus \{p, p'\}$ satisfies $d(o, q_j) + d(p, q_j) < r_0$, we have $\widetilde{T}(p, p') \geq L(\tilde{p}, \tilde{p}')$. We notice that $\widetilde{T}(p, q)$ is the union of $\widetilde{T}(p, p')$ and the subarc of $E(\tilde{o}, \tilde{p}; r_0)$ from \tilde{p}' to \tilde{q} . Therefore, we have $\widetilde{T}(p, q) \geq U(\tilde{p}, \tilde{q}) \geq L(\tilde{p}, \tilde{q})$ (see Lemma 13).

We next prove that there exists for every point $q \in E(o, p; r_0)$ a minimizing geodesic segment $T(p, q)$ such that $\widetilde{T}(p, q) \geq U(\tilde{p}, \tilde{q})$. This is the condition (C1). If $q \notin F_p^{-1}(\widetilde{F}_{\tilde{p}}(\text{Cut}(\tilde{p}) \cap \text{Int}(\widetilde{M}_{\tilde{p}}^+)))$, then it is clear that there exists a minimizing geodesic segment $T(p, q)$ such

that $\tilde{T}(p, q) \geq U(\tilde{p}, \tilde{q})$, since there is the unique minimizing geodesic segment $U(\tilde{p}, \tilde{q}) = L(\tilde{p}, \tilde{q})$ connecting \tilde{p} and \tilde{q} .

If $q \in F_p^{-1}(\tilde{F}_{\tilde{p}}(\text{Cut}(\tilde{p}) \cap \text{Int}(\tilde{M}_{\tilde{p}}^+)))$, then $q \notin E_p(r_0)$ follows from the assumption of Theorems. The assumption (2.1) is used only at this point. Hence, q is not a local maximum point of the distance function to o restricted to $E(o, p; r_0)$, namely $d_{r_0} : E(o, p; r_0) \rightarrow \mathbb{R}$. Therefore, there exists a sequence of points $q_j \in E(o, p; r_0)$ such that $q_j \rightarrow q$ with $d(o, q_j) > d(o, q)$. Since the sequence of minimizing geodesic segments $T(\tilde{p}, \tilde{q}_j)$ and the sequence of curves $\tilde{T}(p, q_j)$ converges to $U(\tilde{p}, \tilde{q})$ (see Lemma 13) and a curve $\tilde{T}(p, q)$, respectively, it follows that there exists a minimizing geodesic segment $T(p, q)$ such that $\tilde{T}(p, q) \geq U(\tilde{p}, \tilde{q})$. \square

Up to this point we have proved (2.2) and (2.3) for all points $q \in M$ with $d(o, q) + d(p, q) \leq r_0$. In order to prove that $M \subset B(o, p; r_0)$, we suppose $M \setminus B(o, p; r_0) \neq \emptyset$ and derive a contradiction.

When we employ $\tilde{M}_{\delta(R)}$, $R > r_0$, as the reference surface of M (see Remark 21) and make the same arguments as in the proofs of Assertions 25 and 26, we have $r_0(R)$ instead of r_0 . We prove $M \subset B(o, p; r_0(R))$ for all $R > r_0$ which contradicts $M \setminus B(o, p; r_0) \neq \emptyset$ because $\delta(R) \rightarrow 0$ as $R \rightarrow \ell$.

Assertion 27. *Suppose that $M \setminus B(o, p; r_0(R)) \neq \emptyset$. Then, there exists an $r_1 > r_0$ such that all points $x \in M$ with $d(o, x) + d(p, x) < r_1$ belong to $F_p^{-1}(\tilde{F}_{\tilde{p}}(\tilde{M}_{\delta(R)\tilde{p}}^+))$.*

Proof. Since $B(o, p; r_0(R))$ is compact, it suffices to find an open set U containing $E(o, p; r_0(R))$ such that every point $q \in U$ and every minimizing geodesic segment $T(p, q)$ have the reference point \tilde{q} and the reference curve $\tilde{T}(p, q)$, respectively. Let $q \in E(o, p; r_0(R))$. As was seen in the proof of Assertion 25, $\theta(\tilde{u})$ is monotone non-increasing as u moves from o to q along $T(o, q)$. We use this fact to determine the location of the reference curve $\tilde{T}(o, q)$ and to study its property.

The complicated case is that the angle of $T(\tilde{o}, \tilde{p}) \cdot (0)$ with $T(\tilde{o}, \tilde{q}) \cdot (0)$ is π . Suppose that $\angle \tilde{p}\tilde{o}\tilde{q} = \pi$. It follows from Assertions 25 and 26 that $\angle poq' = \pi$ for all q' whose reference point is $\tilde{q}' = \tilde{q}$. Hence, we have $q' = q$ if $\tilde{q}' = \tilde{q}$. Moreover, $\angle oup = \angle \tilde{o}\tilde{u}\tilde{p}$ for all $u \in T(o, q)$. If there exists a point $u \in T(o, q)$ such that $\angle oup \neq 0$, then there exists a minimizing geodesic segment $T(p, u)$ such that $\triangle oup$ bounds a totally geodesic 2-dimensional submanifold which is isometric to the comparison triangle domain $\triangle \tilde{o}\tilde{u}\tilde{p}$ in \tilde{M} . This contradicts the present curvature condition. Thus, we obtain $\angle oup = \angle \tilde{o}\tilde{u}\tilde{p} = 0$ for all $u \in$

$T(o, q)$. Therefore, both $T = T(p, o) \cup T(o, q)$ and $\tilde{T} = T(\tilde{p}, \tilde{o}) \cup T(\tilde{o}, \tilde{q})$ are minimizing geodesic segments.

In addition to $\angle \tilde{p}\tilde{o}\tilde{q} = \pi$, suppose $\tilde{q} \in \text{Cut}(\tilde{p})$. From the present curvature assumption, \tilde{q} is not a point conjugate to \tilde{p} along \tilde{T} . Hence, \tilde{q} is not an end cut point of \tilde{p} but branch or regular. In particular, $U(\tilde{p}, \tilde{q})$ is different from \tilde{T} . Since $q \notin \text{Cut}(o)$, Lemmas 14 and 23, there exists a minimizing geodesic segment $T(p, q)$ such that $\tilde{T}(p, q) \geq U(\tilde{p}, \tilde{q})$. Thus, we conclude that q is a cut point of p , since $T(p, q)$ is different from T .

Let W' be a neighborhood of q which is foliated by minimizing geodesic segments from o . Then, from the present curvature assumption, there exist a neighborhood $W \subset W'$ of q and an $\varepsilon > 0$ such that, choosing the appropriate geodesic triangles, $\angle oq'p - \angle \tilde{o}\tilde{q}'\tilde{p} > \varepsilon$ for all points $q' \in W \cap E(o, p; r_0(R))$. Using this property and the same method as in the proof of Assertion 25, we can have a neighborhood V'_q of q such that all points in $V'_q \setminus T(o, q)$ have their reference points in $\text{Int}(\tilde{M}_{\delta(R)\tilde{p}}^+)$.

We next suppose $\tilde{q} \notin \text{Cut}(\tilde{p})$, in addition to $\angle \tilde{p}\tilde{o}\tilde{q} = \pi$. Let $T_e(p, o)$ be the maximal minimizing geodesic from p through o . If q lies in $T_e(p, o)$ but not the endpoint, then the minimizing geodesic segment $T(p, q)$ is unique and $T(p, q) \subset T_e(p, o)$. Even if q is the endpoint of $T_e(p, o)$, then $T_e(p, o) = T(p, o) \cup T(o, q)$ is a minimizing geodesic segment. Therefore, we have $\tilde{T}(p, q) = T(\tilde{p}, \tilde{o}) \cup T(\tilde{o}, \tilde{q})$ as its reference curve, because $\angle oqp = 0$ and (2.3).

Let N be the normal neighborhood around o , namely the domain around o bounded by $\text{Cut}(o)$. Obviously, $T(p, q) = T(p, o) \cup T(o, q) \subset N$. Because $\tilde{q} \notin \text{Cut}(\tilde{p})$, we can have a neighborhood $\tilde{U}_{\tilde{q}}$ of \tilde{q} in $\tilde{M}_{\delta(R)}(\tilde{p})$ such that for all $\tilde{x} \in \tilde{U}_{\tilde{q}} \cap \tilde{M}_{\delta(R)\tilde{p}}^+$, if we write $T(\tilde{p}, \tilde{x})(t) = (r(t), \theta(\tilde{p}) + \theta(t))$ for all $t \in [0, d(\tilde{p}, \tilde{x})]$, then $\exp_o(r(t)(\cos \theta(t)u + \sin \theta(t)v)) \in N$, $0 \leq t \leq d(\tilde{p}, \tilde{x})$. Here $\exp_o : T_o M \rightarrow M$ is the exponential map and v is an arbitrary unit tangent vector such that v is perpendicular to $u := T(o, p)(0)$ in $T_o M$.

We prove that there exists a neighborhood V_q of q so that $V_q \subset N$ and the reference curve $\tilde{T}(p, x)$ is defined for any $x \in V_q$.

Let $x \in N$. Let θ_x denote the angle of $T(o, x)(0)$ with $T(o, p)(0)$. Then θ_x is continuous for $x \in N$. Define a map $\Psi : N \rightarrow \tilde{M}_{\delta(R)\tilde{p}}^+$ by $\Psi(x) = (d(o, x), \theta(\tilde{p}) + \theta_x)$. Since Ψ is continuous, there exists a neighborhood V'_q of q such that $\Psi(V'_q) \subset \tilde{U}_{\tilde{q}} \cap \tilde{M}_{\delta(R)\tilde{p}}^+$.

We claim that all points $x \in V'_q$ have their reference points. Let $x \in V'_q \setminus T_e(o, q)$ where $T_e(o, q)$ denotes the maximal minimizing geodesic from o through q . Since $x \notin T_e(o, q)$, we have $\theta_x \neq \pi$. Let

$r(t)$ and $\theta(t)$ satisfy the equation $T(\tilde{p}, \Psi(x))(t) = (r(t), \theta(\tilde{p}) + \theta(t))$, $0 \leq t \leq d(\tilde{p}, \Psi(x))$. Let $u := T(o, p) \cdot (0)$ and $v_x = T(o, x) \cdot (0)$. Set $v = (v_x - \cos \theta_x u) / \sin \theta_x$ which is the unit tangent vector perpendicular to u and contained in the subspace spanned by $\{u, v_x\}$. Then we define a curve $c(t) = \exp_o(r(t)v(t))$, $0 \leq t \leq d(\tilde{p}, \Psi(x))$, where $v(t) = \cos \theta(t)u + \sin \theta(t)v$. The curve c connects p and x and its length is less than $d(\tilde{p}, \Psi(x))$ because of the curvature condition and the Rauch comparison theorem (see [1]). Therefore, we have $d(p, x) < d(\tilde{p}, \Psi(x))$. Thus, we can define the reference point \tilde{x} of x in $\widetilde{M}_{\delta(R)\tilde{p}}^+$ because $r(\tilde{x}) = r(\Psi(x)) = d(o, x)$, $\theta(\tilde{x}) \leq \theta_x$ and Lemma 11 (1). Since all points $x \in V'_q \cap T_e(o, q)$ are accumulation points of $V'_q \setminus T_e(o, q)$, every point $x \in V'_q$ has its reference point \tilde{x} in $\widetilde{M}_{\delta(R)\tilde{p}}^+$.

From V'_q , we can have a neighborhood V_q of q mentioned above. Suppose for indirect proof that there exists a sequence of points x_j converging to q such that some point $y_j \in T(p, x_j)$ defines the reference curve $\tilde{T}(p, y_j)$ and some point in $T(y_j, x_j)$ close to y_j does not have any reference point. Since those points y_j 's satisfy $d(o, y_j) + d(p, y_j) \geq r_0$, $q \in E(o, p; r_0)$ and Lemma 12 (1), the sequence $d(y_j, x_j)$ goes to zero, and, hence, the sequence of the points y_j converges to q . Thus, $T(y_j, x_j) \subset V'_q$ for a sufficiently large j , contradicting that all points $x \in V'_q$ have their reference points. Therefore, we have the neighborhood V_q as required.

Suppose that $\angle \tilde{p}\tilde{o}\tilde{q} < \pi$. Let $\tilde{U}_{\tilde{q}} \subset \text{Int}(\widetilde{M}_{\delta(R)}^+)$ be a neighborhood of \tilde{q} . Then there exists a neighborhood V'_q of q in M such that $\tilde{F}_{\tilde{p}}^{-1} \circ F_p(V'_q) \subset \tilde{U}_{\tilde{q}}$. As the argument above, we can have a neighborhood V_q of q as required.

Thus, we have found the set $U = \bigcup_{q \in E(o, p; r_0)} V_q$ which is a neighborhood around $E(o, p; r_0(R))$ such that $U \subset F_p^{-1}(\tilde{F}_{\tilde{p}}(\widetilde{M}_{\delta(R)\tilde{p}}^+))$. \square

Assertion 28. *There exists an r_2 with $r_0(R) < r_2 \leq r_1$ such that the condition (C1) is true for any point $q \in E(o, p; r)$, $r_0(R) < r < r_2$.*

Proof. Suppose for indirect proof that (C1) is not true for any $r > r_0(R)$, namely there exists a sequence of $r_j > r_0(R)$ such that r_j converges to $r_0(R)$ and there are no minimizing geodesic segments $T(p, q_j)$ with $\tilde{T}(p, q_j) \geq U(\tilde{p}, \tilde{q}_j)$ for some $q_j \in E(o, p; r_j)$. Suppose without loss of generality that q_j converges to $q_0 \in E(o, p; r_0(R))$. We then have either

$$\tilde{T}(p, q_j) \leq U(\tilde{p}, \tilde{q}_j) \quad \text{or} \quad \tilde{T}(p, q_j) \cap U(\tilde{p}, \tilde{q}_j) \neq \{\tilde{p}, \tilde{q}_j\}.$$

Let $q'_j = T(p, q_j) \cap E(o, p; r_0(R))$. Then, we have $\tilde{T}(p, q'_j) \subset \tilde{T}(p, q_j)$, since $T(p, q'_j) \subset T(p, q_j)$. It follows from the choice of $r_0(R)$ and the condition (C1) that $\tilde{T}(p, q'_j) \geq U(\tilde{p}, \tilde{q}'_j)$. If the first inequality is true, we then have $\tilde{T}(p, q_0) = U(\tilde{p}, \tilde{q}_0)$ as its limit. This contradicts our curvature condition $K_{\delta(R)}$.

If the second situation occurs, we then have the reverse inequality for some point $q' \in T(p, q_j)$ near q_j for sufficiently large j so that

$$\tilde{R}(q', q_j) \leq T(\tilde{q}', \tilde{q}_j)$$

because of Lemma 16 (3) and (4). This contradicts Lemma 19. Therefore, (C1) is true for some $r_2 > r_0(R)$. \square

Assertion 29. *The condition (C2) is satisfied for all $q \in E(o, p; r)$, $r_0(R) < r < r_2$.*

Proof. Let $q \in M$ with $d(o, q) + d(p, q) < r_2$. For convenience, we set $q(t) = T(p, q)(t)$ and $\tilde{q}(t) = \tilde{T}(p, q)(t)$, $0 \leq t \leq d(p, q)$. Let t_0 be the least upper bound of the set of all $t_1 \leq d(p, q)$ so that there exists a minimizing geodesic segment $T(\tilde{p}, \tilde{q}(t))$ with $\tilde{T}(p, q(t)) \geq T(\tilde{p}, \tilde{q}(t))$ for all $t \in (0, t_1)$. Recall that $t_0 > 0$ because of Lemma 19. Suppose for indirect proof that $t_0 < d(p, q)$. If $\tilde{q}(t_0) \notin \text{Cut}(\tilde{p})$, then there exists a positive ε such that $\tilde{T}(p, q(t_0 + t)) \geq T(\tilde{p}, \tilde{q}(t_0 + t))$ for all $t \in (0, \varepsilon)$ because of Lemma 22. This contradicts the choice of t_0 . Suppose $\tilde{q}(t_0) \in \text{Cut}(\tilde{p})$. Since the minimizing geodesic segment $T(p, q(t_0))$ is unique and (C1) is satisfied, we have $\tilde{T}(p, q(t_0)) \geq U(\tilde{p}, \tilde{q}(t_0))$. As is observed in the proof of Lemmas 23 and 24, there exists a positive ε such that $\tilde{T}(p, q(t_0 + t)) \geq U(\tilde{p}, \tilde{q}(t_0 + t))$ for all $t \in (0, \varepsilon)$. This contradicts the choice of t_0 . Hence, it follows that $\tilde{T}(p, q) \geq T(\tilde{p}, \tilde{q})$. \square

Assertions 27 to 29 imply that $M \setminus B(o, p; r_0(R)) \neq \emptyset$ is false when we employ the reference surface $\widetilde{M}_{\delta(R)}$. Since $\delta(R) \rightarrow 0$ as $R \rightarrow \ell$, we conclude that $M \setminus B(o, p; r_0) \neq \emptyset$ is false to the original reference surface of revolution \widetilde{M} ($\delta(\ell) = 0$). This completes the proof of Theorems 5 and 7. \square

The following proposition has been proved in the above argument.

Proposition 30. *Let M and p satisfy the same assumption as in Theorem 5. Then, a point $q \in M$ is a cut point of p if there exists a minimizing geodesic segment $T(p, q)$ such that $\tilde{T}(p, q) \not\geq U(\tilde{p}, \tilde{q})$.*

Proof. As was seen in the proof of Theorem 5, if the reference point \tilde{q} is in $\text{Int}(\widetilde{M}_p^+)$, then there exists a minimizing geodesic T connecting p and

q such that $\tilde{T} \geq U(\tilde{p}, \tilde{q})$. Therefore, we have at least two minimizing geodesics T and $T(p, q)$ connecting p and q . This implies that $q \in \text{Cut}(p)$.

Suppose that $\theta(\tilde{q}) = 0$ or π . Then, as was seen in the proof of Assertion 27, there are two possibilities. One is that $\tilde{T}(p, q) = T(\tilde{p}, \tilde{o}) \cup T(\tilde{o}, \tilde{q})$ and it is a minimizing geodesic segment in \tilde{M} . Then, it follows from the curvature condition that if $q \notin \text{Cut}(p)$, then $\tilde{q} \notin \text{Cut}(\tilde{p})$. Then, $\tilde{T}(p, q) \geq U(\tilde{p}, \tilde{q})$ is true, a contradiction. The other is that a geodesic triangle $\triangle opq$ bounds a totally geodesic 2-dimensional submanifold in M which is isometric to the comparison triangle domain $\triangle \tilde{o}\tilde{p}\tilde{q}$ in \tilde{M} . In this case, for any $\delta > 0$, we regard \tilde{M}_δ as a reference surface of \tilde{M} and M . Then, the reference point $\tilde{q} \in \tilde{M}_\delta^+$ of q belongs to $\text{Int}(\tilde{M}_\delta^+)$ and, moreover, the boundary of the set of the reference points of all points in \tilde{M} . If $q \notin \text{Cut}(p)$, then this contradicts Lemma 16 (1), meaning that $q \in \text{Cut}(p)$. \square

Remark 31. From the proof of Theorems we notice that the assumption $F_p(E(p)) \cap \tilde{F}_{\tilde{p}}(\text{Cut}(\tilde{p}) \cap \text{Int}(\tilde{M}_{\tilde{p}}^+)) = \emptyset$ can be replaced by the following condition: If $q \in F_p^{-1}(\tilde{F}_{\tilde{p}}(\text{Cut}(\tilde{p}) \cap \text{Int}(\tilde{M}_{\tilde{p}}^+)))$, then there exists a minimizing geodesic segment $T(p, q)$ with $\tilde{T}(p, q) \geq U(\tilde{p}, \tilde{q})$.

8. MAXIMUM PERIMETER AND DIAMETER

We have corollaries which are the special version of Corollary 3.

Corollary 32. *Let (M, o) and all points $p \in M$ with $p \neq o$ satisfy the same assumption in Theorem 5. Then every geodesic triangle $\triangle opq$ admits its comparison triangle $\triangle \tilde{o}\tilde{p}\tilde{q}$ in \tilde{M} . In particular, if $\ell < \infty$, we have*

$$d(o, p) + d(p, q) + d(o, q) \leq 2\ell$$

and the diameter of M is less than or equal to ℓ .

Proof. As was seen in the paragraph just before Lemma 12, we have

$$d(o, q) + d(p, q) = d(\tilde{o}, \tilde{q}) + d(\tilde{p}, \tilde{q}) \leq 2\ell - d(\tilde{p}, \tilde{o}) = 2\ell - d(p, o).$$

Therefore, we have $d(o, p) + d(p, q) + d(o, q) \leq 2\ell$.

Let p and q be points in M such that $d(p, q)$ is the diameter of M . It is clear that if $p = o$, then $d(p, q) \leq \ell$. Suppose that $p \neq o$. If \tilde{p} and \tilde{q} in \tilde{M} are the reference points of p and q , respectively, we then have

$$d(\tilde{p}, \tilde{q}) \leq \min\{d(\tilde{p}, \tilde{o}) + d(\tilde{o}, \tilde{q}), d(\tilde{p}, \tilde{o}_1) + d(\tilde{o}_1, \tilde{q})\} \leq \ell,$$

where \tilde{o}_1 is the antipodal point of \tilde{o} in \tilde{M} . Therefore, we have $d(p, q) \leq \ell$. \square

We have the maximum diameter theorem and the maximum perimeter theorem if the assumption (2.1) is extended to the boundary of \widetilde{M}_p^+ .

Corollary 33. *Let (M, o) be a complete pointed Riemannian manifold (M, o) which is referred to a surface of revolution $(\widetilde{M}, \tilde{o})$ with $\ell < \infty$. Assume that all points $p \in M$ with $p \neq o$ satisfy*

$$(8.1) \quad F_p(E(p)) \cap \widetilde{F}_{\tilde{p}}(\text{Cut}(\tilde{p}) \cap \widetilde{M}_{\tilde{p}}^+) = \emptyset.$$

If there exists a pair of points p and q in M such that the perimeter of the geodesic triangle $\triangle opq$ is 2ℓ , then M is isometric to the warped product manifold whose warping function is the radial curvature function of \widetilde{M} . In particular, the same conclusion holds for M if the diameter of M is ℓ .

Proof. Suppose that the perimeter of the geodesic triangle $\triangle opq$ is 2ℓ . From Theorem 5, it has a comparison triangle $\triangle \tilde{o}\tilde{p}\tilde{q}$ in $\widetilde{M}_{\tilde{p}}^+$. We then have $d(\tilde{p}, \tilde{q}) = d(\tilde{p}, \tilde{o}_1) + d(\tilde{o}_1, \tilde{q})$, since

$$\begin{aligned} & d(\tilde{p}, \tilde{q}) + d(\tilde{p}, \tilde{o}) + d(\tilde{o}, \tilde{q}) \\ &= 2\ell \\ &= 2d(\tilde{o}, \tilde{o}_1) \\ &= d(\tilde{p}, \tilde{o}_1) + d(\tilde{o}_1, \tilde{q}) + d(\tilde{p}, \tilde{o}) + d(\tilde{o}, \tilde{q}), \end{aligned}$$

where \tilde{o}_1 is the antipodal point of \tilde{o} in \widetilde{M} . This implies that $U(\tilde{p}, \tilde{q}) = T(\tilde{p}, \tilde{o}_1) \cup T(\tilde{o}_1, \tilde{q})$. From the assumption (8.1) and the condition (C1), there exists a minimizing geodesic segment $T(p, q)$ in M such that $\tilde{T}(p, q) \geq U(\tilde{p}, \tilde{q})$. Thus, we can find a point $o_1 \in T(p, q)$ whose reference point is \tilde{o}_1 . Since $d(o, o_1) = d(\tilde{o}, \tilde{o}_1) = \ell$, The farthest point theorem in [2] concludes our corollary.

Suppose that the diameter of M is ℓ . Let the distance between p and q be ℓ . If $p = o$, then the statement follows from the farthest point theorem (see [2]). Suppose $p \neq o$. As was seen in the proof of Corollary 32, we have

$$\ell = d(\tilde{p}, \tilde{q}) \leq \min\{d(\tilde{p}, \tilde{o}) + d(\tilde{o}, \tilde{q}), d(\tilde{p}, \tilde{o}_1) + d(\tilde{o}_1, \tilde{q})\} \leq \ell.$$

Since

$$d(\tilde{p}, \tilde{o}) + d(\tilde{o}, \tilde{q}) + d(\tilde{p}, \tilde{o}_1) + d(\tilde{o}_1, \tilde{q}) = 2\ell,$$

we have

$$d(\tilde{p}, \tilde{q}) = d(\tilde{p}, \tilde{o}) + d(\tilde{o}, \tilde{q}) = d(\tilde{p}, \tilde{o}_1) + d(\tilde{o}_1, \tilde{q}) = \ell.$$

Thus, the perimeter of the comparison triangle $\triangle \tilde{o}\tilde{p}\tilde{q}$ in \widetilde{M} of $\triangle opq$ is 2ℓ . The maximal perimeter theorem prove the maximal diameter theorem. \square

Remark 34. If the Gauss curvature of the reference surface \widetilde{M} is a positive constant κ , we do not need the assumption (8.1). In fact, as was seen in the proof of Corollary 33, we have $U(\tilde{p}, \tilde{q}) = T(\tilde{p}, \tilde{o}_1) \cup T(\tilde{o}_1, \tilde{q})$ if the perimeter of $\triangle opq$ is 2ℓ . If $d(\tilde{p}, \tilde{q}) < \ell = \pi/\sqrt{\kappa}$, then the minimizing geodesic segment is unique, meaning that $U(\tilde{p}, \tilde{q}) = L(\tilde{p}, \tilde{q})$. This implies that there exists a minimizing geodesic segment $T(p, q)$ in M such that $\widetilde{T}(p, q) \geq T(\tilde{p}, \tilde{q})$ as the limit of the positional relations in $\text{Int}(\widetilde{M}_p^+)$. Thus, we have a point o_1 whose reference point is \tilde{o}_1 . In particular, the diameter of M is ℓ . In the case of $d(\tilde{p}, \tilde{q}) = \ell$, it is clear that the diameter of M is ℓ . Therefore, the maximum diameter theorem states that M is a sphere with constant curvature κ .

Proof of Corollary 4. We first prove that there exists a straight line in \widetilde{M} if there is a straight line in M . Let $T(t)$, $-\infty < t < \infty$, be a straight line in M . Let t_0 be a parameter such that $d(o, T(t_0)) = d(o, T)$. We set $\widetilde{T}(t_1) = (d(o, T(t_1)), 0)$ for all $t_1 \in (-\infty, t_0)$, and $\widetilde{T}(t) = \widetilde{F}_{\widetilde{T}(t_1)}^{-1} \circ F_{T(t_1)}(T(t))$ for any $t \in (t_1, \infty)$. Then it follows from Theorem 5 that $\widetilde{T}(t)$, $t \geq t_1$, is a curve in $\widetilde{M}_{\widetilde{T}(t_1)}^+$ such that $\widetilde{T} \geq T(\widetilde{T}(t_1), \widetilde{T}(t))$ for all $t \geq t_1$. The sequence of minimizing geodesic segments $S_t = T(\widetilde{T}(t_1), \widetilde{T}(t))$ connecting $\widetilde{T}(t_1)$ and $\widetilde{T}(t)$ contains a subsequence S_k converging to a ray S emanating from $\widetilde{T}(t_1)$ as $k \rightarrow \infty$. Let the ray be denoted by $S(t_1)(t)$, $t_1 \leq t$. Then, $d(\tilde{o}, S(t_1)(t_0)) \leq d(o, T)$. From this fact we can find a sequence of rays $S(k)$ converging to a straight line S as $k \rightarrow -\infty$.

It is known that if there is a straight line in \widetilde{M} , then the total curvature of \widetilde{M} is nonpositive. Therefore M has no straight line. If M has at least two ends, then there is a straight line connecting distinct ends. This is impossible because the total curvature of \widetilde{M} is positive. \square

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